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EQUATIONS OF NON-IDEAL MAGNETOHYDRODYNAMICS

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Abstract. In existing derivations[1-4] of fluid equations for strongly magnetized collisional plasmas from kinetic equations, the component of mass velocity perpendicular to the magnetic field is treated as an independent fluid variable with its own equation for temporal variation. This is not in strict accord with the Chapman-Enskog method that forms the basis of these derivations, as the plasma does not relax to a state with arbitrary perpendicular mass velocity in the rapid time scale that includes gyro-motions together with Coulomb collisions. It is shown that this difficulty can be circumvented if the equations for the plasma variables are coupled to the equations for the electromagnetic field, in which displacement current is neglected and quasi-neutrality is assumed. In the hydrodynamic time scale, ideal magnetohydrodynamic equations are obtained. In the next order, the equations include transport fluxes. Although in the latter case, the resulting equations do not differ substantially from those in existing works, they are expressed in forms that more clearly exhibit dynamical consistency. Thus, the generalized Ohm's law is in a form that emphasizes its role for eliminating the electric field from Faraday's law. The electric current density in the $\vec{j} \times \vec{B}$ term of the perpendicular momentum equation is related to the magnetic field through Ampere's law, obviating its determination from the second order kinetic equation, which is cumbersome to solve. The heat flux depends on electric current density similar to thermoelectric effect for metals and semiconductors. In addition to simple plasmas, where comparison with existing works is made, the transport fluxes are also obtained for plasmas with two ion species. The limit of small mass velocity leads to classical transport, wherein the elimination of the electric field from the $\vec{E} \times \vec{B}$ motion causes the local particle flux to depend on boundary conditions.

I. Introduction And Outline

The derivation of fluid equations from kinetic theory of plasmas in a magnetic field has a long history. The early part of the history can be found in the work of Robinson and Bernstein[1]. Most of the work cited there apply strictly to plasmas with short collisional mean-free-path, for which the approach of Chapman and Enskog[2,3] provides a rigorous justification of the closure of the equations in terms of a few fluid variables. The equations of Braginskii[4] for two-temperature plasmas can be considered a culmination of this line of work, and have since been reproduced in books.[5,6,7]. These equations are applicable to a wide variety of plasmas such as industrial plasmas with low temperature, laser plasmas with high density, and space plasmas with large spatial scale.

The existing derivations share a common puzzling feature regarding the fluid velocities perpendicular to the magnetic field. In carrying out the expansion of the distribution function pursuant to the Chapman-Enskog method, the ratio of gyro-radius over scale length is considered to be of the same order as the collisional mean-free-path over scale length, both being adopted as the unique ordering parameter. As a result, the relaxation to equilibrium in the time scales of gyro-motion and collisions yields Maxwellian distributions with arbitrary parallel (to the magnetic field) mass velocity but no perpendicular mass velocity. A non-zero perpendicular mass velocity component $\vec{V}_\perp$ is restored by taking the combination $\vec{E}+\vec{V}_\perp \times \vec{B}/c$ to be small (of the first order), and carrying out the expansion with the distribution function expressed in the variable $\vec{v}' = \vec{v} - \vec{V}$. A result of the derivation is the perpendicular momentum equation [Eq.(4)] giving the time rate of change of $\vec{V}_\perp$. The implication that $\vec{V}_\perp$ is an independent variable characterizing the state of the plasma is at odds with the spirit of the Chapman-Enskog method originally developed for neutral gases. The method identifies gas species densities, temperature, and mass flow as state variables because (1) they are parameters occurring in the leading order distribution functions (Maxwellians), (2) they possess arbitrariness in view of the conservation laws of collisions, which is removed by requiring the absence of higher order corrections to these parameters, and (3) their time variations are determined from the solvability conditions for the first order distribution functions arising from the same conservation laws. Since these features do not strictly apply to $\vec{V}_\perp$, its treatment should be quite distinct from those for the other state variables. The main contribution of the present work is to elucidate how this can be done with a reasonable degree of rigor.

The validity of $\vec{E}+\vec{V}_\perp \times \vec{B}/c$ remaining small would be in doubt if the electric field is prescribed, or if the displacement current is kept, as it cannot be guaranteed at face value by the independent time variations of $\vec{V}_\perp$ and the electric field. We recognize that this objection can be removed if the goal is to produce a dynamically consistent set of equations for both the plasma fluid variables and the electromagnetic field, provided that the displacement current is neglected and quasi-neutrality is assumed. Without such stipulation, as is the case for the cited works, the objection would remain. A crucial step taken in the present approach is to subject both the electric field and the perpendicular mass flow to expansions in the ordering parameter so that $\vec{E} = \vec{E}^{(-1)} + \vec{E}^{(0)} + \cdots$, $\vec{V}_\perp = \vec{V}_\perp^{(0)} + \vec{V}_\perp^{(1)} + \cdots$. It is found that quasi-steady state in the time scale of collisions and gyromotions occurs if

$$\vec{E}^{(-1)} + \frac{1}{c}\vec{V}_\perp^{(0)} \times \vec{B} = 0$$

In the time scale slower by one order, an expression for $\vec{E}^{(0)} + \vec{V}_\perp^{(1)} \times \vec{B}/c$ can be found in terms of fluid variables that include the first order current density $\vec{j}^{(1)}$. The smallness of $\vec{E}+\vec{V}_\perp \times \vec{B}/c$ is thus formally guaranteed.

With the neglect of the displacement current, there is no equation for the time variation of the electric field, which must therefore be eliminated in terms of other variables. The expressions for $\vec{E}+\vec{V}_\perp \times \vec{B}/c$ in the two leading orders serve this purpose. In existing works, the first-order current density is calculated from the first-order distribution function, resulting in a relation of the form

$$\vec{j}^{(1)} = L\left\{\vec{E}_\parallel^{(0)}, \vec{E}_\perp^{(0)} + \vec{V}_\perp^{(1)} \times \vec{B}/c, \nabla n_a, \nabla T\right\}$$

known as the generalized Ohm's law, where n_a is the density of plasma species labeled a, T is the common temperature, and L denotes a linear function. Solving the relation for $\vec{E}_\parallel^{(0)}$ and

$\vec{E}_\perp^{(0)} + \vec{V}_\perp^{(1)} \times \vec{B}/c$ and combining with the equation $\vec{E}^{(-1)} + \vec{V}_\perp^{(0)} \times \vec{B}/c = 0$ leads to an equation of the form.

$$\vec{E} + \frac{\vec{V}}{c} \times \vec{B} = L^{-1}\left\{\vec{j}^{(1)}, \nabla n_a, \nabla T\right\}$$

with another linear function formally written as L^{-1}. The above equation can be used to eliminate $\vec{E}$ from Faraday's law and other fluid equations. But the results contain the variables $\vec{j}^{(1)}$ and $\vec{V}_\perp$.

The first order current density $\vec{j}^{(1)}$ can be eliminated in favor of the magnetic field, considered an independent variable, if the replacement

$$\vec{j}^{(1)} = c\nabla \times \vec{B}/4\pi$$

is made to satisfy Ampere's law. The perpendicular mass flow $\vec{V}_\perp$ can be elevated to be an independent plasma parameter with time variation given by the perpendicular momentum equation. It is important to note that the $\vec{j} \times \vec{B}$ term in this equation requires the second order current density $\vec{j}^{(2)}$ if the momentum equation is to be accurate in the transport time scale. Also there is no other equation for $\vec{j}^{(2)}$ from the plasma dynamics, because, even if we were to solve for the second order distribution functions-which is not necessary-we would have found $\vec{j}^{(2)}$ to satisfy the first order perpendicular momentum equation. Again we can appeal to Ampere's law to eliminate $\vec{j}^{(2)}$ in favor of the magnetic field, provided we sum the perpendicular momentum equations in the two leading order time scales, and make the replacement

$$\vec{j}^{(1)} + \vec{j}^{(2)} = c\nabla \times \vec{B}/4\pi$$

in the $\vec{j} \times \vec{B}$ term. It is the possibility of making this replacement that allows the determination of the time variation of $\vec{V}_\perp$, which is in a manner different than those for densities, temperature, and parallel mass flow. Failure to acknowledge this consideration in the cited works is a source of confusion.

Replacing $\vec{j}^{(1)}$ by $\vec{j}^{(1)} + \vec{j}^{(2)}$ in the generalized Ohm's law does not impact its accuracy. The price paid is just the incursion of an extraneous order. However, the generalized Ohm's law should not be used to determine the current density in terms of the electric field for substitution in the $\vec{j} \times \vec{B}$ term of the momentum equation. This point is easily lost sight of in existing works, which do not emphasize that the role of the generalized Ohm's law is to eliminate the electric field. Indeed, the point is completely lost when the generalized Ohm's law is defined as one that gives the time rate of change of the current density, as in Ref. (9).

In this paper, we present a derivation of the fluid equations for single-temperature plasmas following the approach outlined, which gives rise to a consistent set of dynamical equations with the densities of plasma species, the common temperature, the mass velocity, and the magnetic field as independent variables. The current density is obtained from the curl of the magnetic field; the electric field is eliminated from Faraday's law and other equations using the generalized Ohm's law. In the hydrodynamic time scale, the equations of ideal magnetohydrodynamics are obtained. Including the time scale in the next order, the equations describe plasma transport. The heat flux and viscous stress tensors are evaluated for simple plasmas, in the subsidiary limit when the ratio of gyroradius to collisional mean free path is small. They are equivalent to those in Ref.(4), although in the present work, the heat flux is expressed in a form that depends on the current density, similar to the Peltier effect for metals and semiconductors. Also, we include corrections of the order of the square root of electron-ion mass ratio in the ion perpendicular thermal conductivity and parallel viscosity, which are of the same order of magnitude as the electron contributions to the same transport coefficients. The generalized Ohm's law obtained for simple plasmas can also be retrieved from various equations of Ref.(4). The transport fluxes and generalized Ohm's law are also evaluated for plasmas with two ion species, which has applications to D-T plasmas and plasmas with impurity ions, and which exhibits features such as inter-species diffusion that do not occur for simple plasmas.

The term classical transport[8] customarily refers to the situation where the mass velocity is much less than the ion thermal velocity. It can be deduced from the fluid equations of the present theory by allowing the mass velocity to become first-order small. Since in this limit the pressure gradient is balanced by the $\vec{j}\times\vec{B}$ force, there is no equation for the time variation of the perpendicular mass velocity. As a result, the electric field cannot be eliminated using the generalized Ohm's law, because it involves the perpendicular velocity. We treat the constraint of force balance at all times by requiring the time derivative of the force balance equation to be satisfied. The resulting equation determines an electric field that depends on boundary conditions, that is imparted to the local particle flux.

We organize the paper as follows. The next section presents the exact moment equations and the equations for the electromagnetic field, which remain valid in all orders of time scale. In Section 3, the formal expansion procedure is carried out to the zeroth order, which is taken to be the time scale of hydrodynamics. In Section 4, the expansion procedure is continued to the first order, producing the linearized kinetic equation and expressions for the transport fluxes. The procedure is specialized to simple plasmas in Section 5, solving the resulting equations recovers most results of Ref. (4). Section 6 deals with plasmas with two ion species, presenting the appropriate forms of the transport fluxes and Ohm's law. Section 7 discusses the classical transport limit when the mass velocity is considered to be small. It is followed by a brief concluding section. Appendix A provides the conditions for the justifications for quasi-neutrality and the neglect of displacement current. Appendix B gives the details for calculating the transport fluxes using expansions in Sonine polynomials, and presents sample results of the calculations.

II. The Moment Equations

We begin with the kinetic equation for a multi-component plasma. For the charged particle species labeled a, of mass m_a and charge e_a, in an electric field $\vec{E}$ and magnetic field $\vec{B}$, the distribution function f_a satisfies the equation

$$\frac{\partial f_a}{\partial t}+\vec{\upsilon}\cdot\nabla f_a+\frac{e_a}{m_a}\left(\vec{E}+\frac{\vec{\upsilon}}{c}\times\vec{B}\right)\cdot\frac{\partial f_a}{\partial\vec{\upsilon}}=\sum_b C_{ab}\left(f_a,f_b\right) \tag{1}$$

where C_{ab} is the Fokker-Planck operator for Coulomb collisions. In the approach of Chapman and Enskog, the fluid equations are obtained in different stages or frequency scales through expansion of the distribution function, in the ratio of collisional mean-free-path to scale length for neutral gases, and also in the ratio of gyroradius to scale length for strongly magnetized plasmas. They are the conditions of solvability of the kinetic equations in the different stages. A convenient way to obtain these fluid equations is to use the exact moment equations of Eq.(1), and evaluate the moments using the approximate distribution functions in the different stages as the solvability conditions are called for. The moment equations follow from the conservation laws of the collisions, and they are the four equations below:

$$\frac{d\rho}{dt}+\rho\nabla\cdot\vec{V}=0 \quad , \tag{2}$$

$$\frac{dn_a}{dt}+n_a\nabla\cdot\vec{V}+\nabla\cdot n_a\vec{u}_a=0 \quad , \tag{3}$$

$$\rho\frac{d\vec{V}}{dt}+\nabla p+\nabla\cdot\vec{\pi}-\frac{1}{c}\vec{j}\times\vec{B}=0 \quad , \tag{4}$$

$$\frac{3}{2}n\frac{dT}{dt}+p\nabla\cdot\vec{V}+\vec{\pi}:\nabla\vec{V}-\frac{3}{2}T\nabla\cdot\sum_a n_a\vec{u}_a+\nabla\cdot\vec{q}-\vec{j}\cdot\left(\vec{E}+\frac{\vec{V}}{c}\times\vec{B}\right)=0 \quad . \tag{5}$$

Combining Eq. (3) and Eq. (5) gives the following somewhat simpler equation than Eq. (5):

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right)+\nabla\cdot\left(\frac{3}{2}nT\vec{V}\right)+p\nabla\cdot\vec{V}+\vec{\pi}:\nabla\vec{V}+\nabla\cdot\vec{q}-\vec{j}\cdot\left(\vec{E}+\frac{\vec{V}}{c}\times\vec{B}\right)=0 \tag{6}$$

In these equations, n_a is the species density, $n=\sum n_a$, $\rho_a=m_a n_a$, $\rho=\sum\rho_a$, $\vec{V}_a$ is the species velocity, $\vec{V}$ is the mass velocity given by $\rho\vec{V}=\sum\rho_a\vec{V}_a$, $d/dt=\partial/\partial t+\vec{V}\cdot\nabla$, $\vec{u}_a=\vec{V}_a-\vec{V}$, $\vec{j}=\sum_a e_a n_a\vec{V}_a$, T is the common temperature defined by

$$\frac{3}{2}\sum_a n_a T=\sum_a\int d^3\upsilon\frac{1}{2}m_a\left(\vec{\upsilon}-\vec{V}\right)^2 f_a \tag{7}$$

$p_a=n_aT$, $p=\sum_a p_a$, $\vec{q}_a$ is the species heat flux (relative to the mass velocity) defined by

$$\vec{q}_a = \int d^3\upsilon \frac{1}{2} m_a \left(\vec{\upsilon} - \vec{V}\right)^2 \left(\vec{\upsilon} - \vec{V}\right) f_a \tag{8}$$

$\vec{q} = \sum_a \vec{q}_a$, $\vec{\vec{\pi}}_a$ is the species viscous stress tensor defined by

$$\vec{\vec{\pi}}_a = \int d^3\upsilon m_a \left[\left(\vec{\upsilon} - \vec{V}\right)\left(\vec{\upsilon} - \vec{V}\right) - \frac{1}{3}\left(\vec{\upsilon} - \vec{V}\right)^2 \vec{\vec{I}} \right] f_a \tag{9}$$

and $\vec{\vec{\pi}} = \sum \vec{\vec{\pi}}_a$. Also, the quasi-neutrality condition $\sum e_a n_a = 0$ has been assumed. Using these equations, the entropy density as defined by $s = \sum_a s_a \doteq \sum_a n_a \ell n\left(T^{3/2}/n_a\right)$ can be shown to satisfy the equation

$$\frac{ds}{dt} + s\nabla\cdot\vec{V} + \nabla\cdot\sum_a \left[s_s \vec{u}_a + \frac{1}{T}\left(\vec{q}_a - \frac{5}{2} p_a \vec{u}_a \right) \right] = \theta \quad , \tag{10}$$

where the entropy density production rate θ is given from

$$T\theta = \vec{j}\cdot\left(\vec{E} + \frac{\vec{V}}{c} \times \vec{B} \right) - \sum_a \left[\vec{u}_a \cdot \nabla p_a + \left(\vec{q}_a - \frac{5}{2} p_a \vec{u}_a \right) \cdot \nabla \ell n T + \vec{\vec{\pi}}_a : \nabla\vec{V} \right] \quad . \tag{11}$$

For the electromagnetic field, neglecting the displacement current, Ampere's law and Faraday's law apply:

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} \tag{12}$$

$$\frac{\partial\vec{B}}{\partial t} = -c\nabla\times\vec{E} \tag{13}$$

The need to impose quasi-neutrality and to neglect displacement current arises from the assumed restriction to low frequency nonrelativistic motions and scale length much longer than Debye length. Details are given in Appendix A. A closed description of the plasma dynamics using only fluid variables will be achieved if the independent variables are identified and all moments are expressed in terms of these variables. The form of the moment equations suggests that the plasma variables can be taken to be the species densities, the mass velocity, and the common temperature. The magnetic field is also an independent variable. The electric current density is just given by its curl. However, with the neglect of displacement current, there is no equation for the time evolution for the electric field, which therefore should be considered as a dependent variable. The quantities to be expressed in terms of the variables $n_a, \vec{V}, T$ and $\vec{B}$ are then $\vec{u}_a, \vec{q}, \vec{\vec{\pi}}$ and $\vec{E}$. These tasks will be accomplished by an expansion of the distribution function.

III. Ideal MHD: Hydrodynamic Stage

To implement the Chapman-Enskog approach starting from Eq. (1), we consider both collisional mean-free-path λ and gyro-radius ρ_B to be small compared to the scale length ℓ, and introduce the ordering parameter δ so that

$$\rho_B/\ell \sim \lambda/\ell \sim \delta . \tag{14}$$

The distribution function is expanded in this parameter:

$$f_a = f_{a0} + f_{a1} + \cdots \tag{15}$$

In contrast to existing works, the electric field is also expanded:

$$\vec{E} = \vec{E}^{(-1)} + \vec{E}^{(0)} + \cdots \tag{16}$$

with $E_{\parallel}^{(-1)} = 0$. The leading terms of the perpendicular and parallel components of the electric field are assigned the orderings

$$\frac{cE_{\perp}^{(-1)}}{B\bar{\upsilon}} \sim \delta^0 \quad \frac{eE_{\parallel}^{(0)}\ell}{T} \sim \delta^0 \quad , \tag{17}$$

where $\bar{\upsilon}$ is the thermal velocity of a typical plasma species. Together, they imply $E_{\parallel}^{(0)} \big/ E_{\perp}^{(-1)} \sim \delta$. The ordering of $E_{\perp}^{(-1)}$ ensures that the perpendicular mass velocity is of the order of the thermal velocity, thus distinguishing the present work from cases commonly referred to as classical transport[8], in which the ratio of mass velocity to thermal velocity is of the order of gyroradius over scale length. It is shown in Section 7 that classical transport can be recovered from the present approach.

Introducing the hydrodynamic frequency scale $\omega_0 \sim \bar{\upsilon}/\ell$, the time derivative in Eq. (1) is formally expanded as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^{(0)}} + \frac{\partial}{\partial t^{(1)}} + \cdots \quad , \tag{18}$$

with $t^{(n)}$ corresponding to the frequency scale $\delta^n \omega_0$.

Assuming that quasi-steady state is reached in the frequency scale $\delta^{-1}\omega_0$, the kinetic equation in this scale is

$$\frac{e_a}{m_a}\left(\vec{E}^{(-1)} + \frac{\vec{\upsilon}}{c} \times \vec{B}\right) \cdot \frac{\partial f_{a0}}{\partial \vec{\upsilon}} = \sum_b C_{ab}\left(f_{a0}, f_{b0}\right) \quad . \tag{19}$$

It has the general solution

$$f_{a0} = n_a \left(\frac{m_a}{2\pi T} \right)^{3/2} \exp\left[-m_a \left(\vec{\upsilon} - \vec{V}^{(0)} \right)^2 \Big/ 2T \right] \tag{20}$$

in which

$$\vec{V}^{(0)} = V_\parallel \hat{b} + \vec{V}_\perp^{(0)} \tag{21}$$

$$\vec{V}_\perp^{(0)} = c \frac{\vec{E}^{(-1)} \times \hat{b}}{B} \tag{22}$$

with $\hat{b} = \vec{B}/B$, and $n_a, T, V_\parallel$ are arbitrary functions of space and time. No superscript is attached to $V_\parallel$ because, unlike $\vec{V}_\perp^{(0)}$, it is an independent variable. The variables $n_a, V_\parallel$ can be taken to be the species density and parallel mass velocity respectively. $\vec{V}_\perp^{(0)}$ is the zeroth-order perpendicular mass velocity. T is the temperature correct to the first order. It is not to all orders in view of the appearance of $\vec{V}^{(0)}$ in f_{a0} rather than the exact mass velocity. But we shall not need to redefine temperature to include a correction to T as the fluid equations that will be obtained are accurate only through the first order frequency scale $\delta\omega_0$. Note that the perpendicular mass velocity is not an independent variable at this stage. In this order, all species move with the same velocity, so that the current density $\vec{j}^{(0)}$ is zero.

The time variation of the independent variables $n_a, T, V_\parallel$ in the zeroth-order frequency scale ω_0 are obtained from the kinetic equation in the same order, which is

$$\frac{\partial f_{a0}}{\partial t^{(0)}} + \vec{\upsilon} \cdot \nabla f_{a0} + \frac{e_a}{m_a} \vec{E}^{(0)} \cdot \frac{\partial f_{a0}}{\partial \vec{\upsilon}} + \frac{e_a}{m_a} \left(\vec{E}^{(-1)} + \frac{\vec{\upsilon}}{c} \times \vec{B} \right) \cdot \frac{\partial f_{a1}}{\partial \vec{\upsilon}} = \sum_b \left[C_{ab}\left(f_{a1}, f_{b0} \right) + C_{ab}\left(f_{a0}, f_{b1} \right) \right] \tag{23}$$

It proves convenient for discussions in later sections to transform to the variable $\vec{\upsilon}'$ defined by

$$\vec{\upsilon}' = \vec{\upsilon} - \vec{V}^{(0)} \quad . \tag{24}$$

The resulting equation is

$$\frac{df_{a0}}{dt^{(0)}} + \vec{\upsilon}' \cdot \nabla f_{a0} + \left(\frac{e_a}{m_a} \vec{E}^{(0)} - \frac{d\vec{V}^{(0)}}{dt^{(0)}} \right) \cdot \frac{\partial f_{a0}}{\partial \vec{\upsilon}'} - \left(\vec{\upsilon}' \cdot \nabla \vec{V}^{(0)} \right) \cdot \frac{\partial f_{a0}}{\partial \vec{\upsilon}'} + \frac{e_a}{m_a c} \vec{\upsilon}' \times \vec{B} \cdot \frac{\partial f_{a1}}{\partial \vec{\upsilon}'} = \sum_b \left[C_{ab}\left(f_{a1}, f_{b0} \right) + C_{ab}\left(f_{a0}, f_{b1} \right) \right] \tag{25}$$

where $d/dt^{(0)} = \partial/\partial t^{(0)} + \vec{V}^{(0)} \cdot \nabla$, and C_{ab} takes the same form in the $\vec{\upsilon}'$ as in the $\vec{\upsilon}$ variables because of Galilean invariance.

As an equation for f_{a1}, it needs to satisfy the solvability conditions obtained by performing on it the operations $\int d^3\upsilon'$, $\sum_a \int d^3\upsilon' m_a \upsilon'_\parallel$, $\sum_a \int d^3\upsilon' m_a \upsilon'^2/2$, which annihilate the terms involving f_{a1} on both sides of the equation. The following equations result:

$$\frac{dn_a}{dt^{(0)}} + n_a \nabla \cdot \vec{V}^{(0)} = 0 \tag{26}$$

$$\rho \hat{b} \cdot \frac{d\vec{V}^{(0)}}{dt^{(0)}} + \hat{b} \cdot \nabla p = 0 \tag{27}$$

$$\frac{3}{2} n \frac{dT}{dt^{(0)}} + p \nabla \cdot \vec{V}^{(0)} = 0 \quad . \tag{28}$$

If the electric and magnetic fields are considered to be external, or if the field equations are ignored, as in many existing works, these equations would, together with Eq. (22) for the perpendicular mass velocity, provide a closed fluid description of the plasma. Otherwise, Ampere and Faraday's laws must be included. Using the estimate $j_\perp^{(1)} = \sum_s e_a \int d^3\upsilon' \vec{\upsilon}'_\perp f_{a1} \sim \delta n e \bar{\upsilon}$ for the first-order current density and Eq. (14), we find

$$\frac{4\pi j_\perp^{(1)}}{c \nabla \times B} \sim \frac{4\pi n e \bar{\upsilon} \rho_B}{cB} \sim \frac{8\pi p}{B^2} \quad . \tag{29}$$

It is then necessary to use $\vec{j}^{(1)}$ in Ampere's law if the plasma beta is not a small parameter, so that

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}^{(1)} \quad . \tag{30}$$

It turns out to be unnecessary to solve for f_{a1} in order to determine $\vec{j}^{(1)}$, because it can be found by performing the operation $\sum_a \int d^3\upsilon' m_a \vec{\upsilon}'$ on Eq. (25), which yields the momentum equation

$$\rho \frac{d\vec{V}^{(0)}}{dt^{(0)}} + \nabla p - \frac{1}{c} \vec{j}^{(1)} \times \vec{B} = 0 \quad . \tag{31}$$

This is an equation for $\vec{j}_\perp^{(1)}$ in so far as $\vec{V}_\perp^{(0)}$ and its time derivative are determined by Eq. (22).

Faraday's law takes the form

$$\frac{\partial \vec{B}}{\partial t^{(0)}} = -c \nabla \times \vec{E}^{(-1)} \quad , \tag{32}$$

where the implied variation in the frequency scale ω_0 is justified when we rewrite Eq. (22) as

$$\vec{E}^{(-1)} + \frac{\vec{V}^{(0)}}{c} \times B = 0 \tag{33}$$

and use it to eliminate $\vec{E}^{(-1)}$ from Eq. (32). (Recall that $E_{\parallel}^{(-1)} = 0$.)

This elimination is necessary because there is no dynamic equation for the electric field when displacement current is neglected. But it means that $\vec{V}_{\perp}^{(0)}$ can no longer be determined from Eq. (22). Instead, it can be considered an independent variable along with the densities, temperature, parallel mass velocity, and magnetic field. Its time variation is now given by Eq. (31), which is no longer considered to be an equation for $\vec{j}_{\perp}^{(1)}$. Rather, $\vec{j}_{\perp}^{(1)}$ is defined from Eq. (30) in terms of $\vec{B}$. The parallel component $j_{\parallel}^{(1)}$ plays no role in the dynamics. Dropping the superscripts in Eqs. (26), (28), (30–32) and (33), it is seen that they are the equations of ideal magnetohydrodynamics.

IV. Transport Stage

To obtain the fluid equations accurate to the first order frequency scale $\delta\omega_0$, it is necessary to determine f_{a1}, which satisfies Eq. (25). Using Eqs. (26–28) to replace the time derivatives, and for convenience introducing f'_{a1} through

$$f_{a1} = \frac{m_a}{T}\vec{\upsilon}' \cdot \vec{V}^{(1)} f_{a0} + f'_{a1} \tag{34}$$

so as to remove the first order mass velocity, Eq. (25) can be conveniently decomposed into four equations so that f'_{a1} is the sum of their solutions. The four consist of two for the gyro-phase averaged part describing parallel transport:

$$\frac{\vec{\upsilon}'_{\parallel}}{p_a} \cdot \left[\nabla p_a - n_a e_a \vec{E}^{(0)} - \frac{\rho_a}{\rho}\nabla p + \left(\frac{m_a \upsilon'^2}{2T} - \frac{5}{2}\right) n_a \nabla T \right] f_{a0} = \sum_b [C_{ab}\left(f'_{a1}, f_{b0}\right) + C_{ab}\left(f_{a0}, f'_{b1}\right)] \tag{35}$$

$$\frac{m_a}{T}\left\langle \vec{\upsilon}'\vec{\upsilon}' - \frac{1}{3}\upsilon'^2 \vec{I} \right\rangle : \nabla \vec{V}^{(0)} f_{a0} = \sum_b [C_{ab}\left(f'_{a1}, f_{b0}\right) + C_{ab}\left(f_{a0}, f'_{b1}\right)] \tag{36}$$

where the brackets $\langle \; \rangle$ denote gyro-phase average, and two for the gyro-phase dependent part describing perpendicular transport:

$$\begin{aligned} &\frac{\vec{\upsilon}'_{\perp}}{p_a} \cdot \left[\nabla p_a - n_a e_a \vec{E}'^{(0)}_{\perp} - \frac{\rho_a}{\rho}(\nabla p - \vec{j}^{(1)} \times \vec{B}) + \left(\frac{m_a \upsilon'^2}{2T} - \frac{5}{2}\right) n_a \nabla T \right] f_{a0} + \frac{e_a}{m_a c}\vec{\upsilon}' \times \vec{B} \cdot \frac{\partial f'_{a1}}{\partial \vec{\upsilon}'} \\ &= \sum_b [C_{ab}\left(f'_{a1}, f_{b0}\right) + C_{ab}\left(f_{a0}, f'_{b1}\right)] \end{aligned} \tag{37}$$

$$\frac{m_a}{T}\left[\vec{\upsilon}'\vec{\upsilon}' - \left\langle\vec{\upsilon}'\vec{\upsilon}'\right\rangle\right]\cdot\nabla\vec{V}^{(0)}f_{a0} + \frac{e_a}{m_a c}\vec{\upsilon}'\times\vec{B}\cdot\frac{\partial f'_{a1}}{\partial\vec{\upsilon}'} = \sum_b\left[C_{ab}\left(f'_{a1},f_{b0}\right) + C_{ab}\left(f_{a0},f'_{b1}\right)\right] \tag{38}$$

where

$$\vec{E}'^{(0)}_{\perp} = \vec{E}^{(0)}_{\perp} + \frac{\vec{V}^{(1)}}{c}\times\vec{B} \tag{39}$$

These solutions are subject to the conditions

$$\int d^3\upsilon' f'_{a1} = 0 \tag{40}$$

$$\sum_a\int d^3\upsilon' m_a\vec{\upsilon}' f'_{a1} = 0 \tag{41}$$

$$\sum_a\int d^3\upsilon'\frac{1}{2}m_a\upsilon'^2 f'_{a1} = 0 \quad . \tag{42}$$

Equations (40) and (42) and the parallel component of Eq. (41) render the solutions unique and identify the parameters n_a, T and $V_\parallel$ as densities, temperature and parallel mass velocity. The perpendicular component of Eq. (41) is required because of the transformation in Eq. (34) that removes the first-order perpendicular mass velocity $\vec{V}_\perp^{(1)}$ from f'_{a1}.

Time derivatives of the fluid variables in the first order frequency scale $\delta\omega_0$ are obtained from the kinetic equation in the same scale, which is a continuation of Eq. (23) to the next order. It reads

$$\begin{aligned}&\frac{\partial f_{a0}}{\partial t^{(1)}} + \frac{\partial f_{a1}}{\partial t^{(0)}} + \vec{\upsilon}\cdot\nabla f_{a1} + \frac{e_a}{m_a}\vec{E}^{(1)}\cdot\frac{\partial f_{a0}}{\partial\vec{\upsilon}} + \frac{e_a}{m_a}\vec{E}^{(0)}\cdot\frac{\partial f_{a1}}{\partial\vec{\upsilon}} + \frac{e_a}{m_a}\left(\vec{E}^{(-1)} + \frac{\vec{\upsilon}}{c}\times\vec{B}\right)\cdot\frac{\partial f_{a2}}{\partial\vec{\upsilon}} \\ &= \sum_b\left[C_{ab}\left(f_{a2},f_{b0}\right) + C_{ab}\left(f_{a1},f_{b1}\right) + C_{ab}\left(f_{a0},f_{b2}\right)\right]\end{aligned} \quad . \tag{43}$$

Instead of transforming this equation to the variable $\vec{\upsilon}'$ and applying the solvability conditions by performing the operations $\int d^3\upsilon', \sum\int d^3\upsilon' m_a\vec{\upsilon}_\parallel, \sum\int d^3\upsilon' m_a\upsilon'^2/2$, it proves more convenient to regard the time derivatives of the fluid variables as given by the moment equations of Section II, and evaluate the moments occurring in these equations accurate to the first order by using the distribution functions f_{a0} and f_{a1}. The results are expressions of transport fluxes in terms of integrals involving f_{a0} and f'_{a1}, which are stated in the following.

To begin with, dropping the superscripts that indicate orders, the stress tensor is found from

$$\vec{\pi} = \sum_a \vec{\pi}_a = \sum_a \int d^3\upsilon' m_a \left(\vec{\upsilon}'\vec{\upsilon}' - \frac{1}{3}\upsilon'^2 \vec{I} \right) f'_{a1} \tag{44}$$

and the heat flux from

$$\vec{q} = \sum_a \vec{q}_a = \sum_b \int d^3\upsilon' \frac{1}{2} m_a \upsilon'^2 f'_{a1} \quad . \tag{45}$$

With first order accuracy, the individual velocity $\vec{V}_a$ is the sum of the mass velocity $\vec{V}$ and the deviation from the mass velocity $\vec{u}_a$:

$$\vec{V}_a = \vec{V} + \vec{u}_a \tag{46}$$

with

$$\vec{u}_a = \frac{1}{n_a} \int d^3\upsilon' \vec{\upsilon}' f'_{a1} \quad . \tag{47}$$

The constraint on the perpendicular component $\vec{u}_{\perp a}$ coming from the perpendicular component of Eq. (41) gives rise to an expression for $\vec{E}'^{(0)}_\perp$ in terms of the fluid variables including the first-order current density $\vec{j}^{(1)}$. The parallel component of Eq. (41), on the other hand, only eliminates the indeterminacy (a term corresponding to a common mass velocity) in the solution, and does not determine $\vec{E}^{(0)}_\parallel$. Instead, we can evaluate the parallel current $j^{(1)}_\parallel$ from

$$j^{(1)}_\parallel = \sum_a n_a e_a u_{\parallel a} \tag{48}$$

and solve for $\vec{E}^{(0)}_\parallel$ in terms of $j^{(1)}_\parallel$ and other fluid variables. The zeroth order electric field $\vec{E}^{(0)}$ thus determined can be combined with Eq. (33) for the lower order $\vec{E}^{(-1)}$ to yield an equation of the form

$$\vec{E} + \frac{\vec{V}}{c} \times \vec{B} = \textit{function of } n_a, T, \nabla n_a, \nabla T, \vec{j} \tag{49}$$

where $\vec{E} = \vec{E}^{(-1)} + \vec{E}^{(0)}$ and $\vec{V} = \vec{V}^{(0)} + \vec{V}^{(1)}$. This equation can be regarded as a generalized Ohm's law.

One might be inclined to express $\vec{j}$ in terms of $\vec{E}$ instead.[1,3] It would then be tempting to regard the current density in the $\vec{j} \times \vec{B}$ term in Eq. (4) to be given by the same expression. But this would be in error because, when accuracy to the frequency scale $\delta\omega_0$ is required, the second order current $\vec{j}^{(2)}$ enters in the $\vec{j} \times \vec{B}$ term. Indeed, when perpendicular mass velocity is chosen as an independent fluid variable, the first-order form of Eq. (2) determines $\vec{j}^{(2)}$, just as the zeroth order form determines $\vec{j}^{(1)}$. On the other hand, if $\vec{j}$ is equated to $c\nabla\times\vec{B}/4\pi$ through the second order, and $\vec{B}$ is considered to be the only independent field variable, Eq. (2) remains valid to the first order. A dynamically consistent set of equations accurate to the frequency scale $\delta\omega_0$ consists of Eqs. (2–4), (12), (13) and (49), together with fluxes determined from Eqs. (44), (45) and (47). They are the equations of non-ideal magnetohydrodynamics.

V. Simple Plasma

A simple plasma contains only one ion species, with charge Z_i and mass m_i. In the relevant equations of the last section in this case, the index a is either e or i. The solution of the transport problem described in this section is also based on treating $\sqrt{m_e/m_i}$ and the ratio of collision frequency to gyro-frequency for both electron and ion to be small. As a result, various quantities acquire diverge magnitudes, causing time variations in many different frequency scales, and creating the problem of deciding what terms to keep for a consistent description in a given time scale. Instead of resolving the time scales, the expediency of simply keeping terms of two lowest orders in each of the fluid variables based on the approximations will be adopted.

A. Parallel velocities and heat fluxes

For Eq. (35), which determines parallel velocities and heat flux, the distribution functions f'_{e1} and f'_{i1} are both of the same order λ/ℓ, the ratio of mean-free-path over parallel gradient length. This gives rise to the estimates from Eqs. (45) and (47) that $u_{\|e}\sim\left(m_e/m_i\right)^{-1/2}\left(\lambda/\ell\right)\bar{\upsilon}_i,\quad q_{\|i}\sim p\left(\lambda/\ell\right)\bar{\upsilon}_i,\quad q_{\|e}\sim\left(m_e/m_i\right)^{-1/2}q_{\|i}$. The condition $\rho_e u_{\|e}+\rho_i u_{\|i}=0$, which follows from the parallel component of Eq. (41), then leads to $u_{\|i}\sim\sqrt{m_e/m_i}\,\bar{\upsilon}_i$, two orders in $\sqrt{m_e/m_i}$ smaller than $u_{\|e}$. If ion parallel heat flux $q_{\|i}$ is included in the temperature equation [Eq. (5)], it is necessary to calculate the first two leading orders of $q_{\|e}$ to maintain a consistent time scale. The electron version of Eq. (35) with the required accuracy is

$$\frac{\vec{\upsilon}'_{\|}}{p_e}\cdot\left[\nabla p_e+n_e e\vec{E}^{(0)}+\left(\frac{m_e\upsilon'^2}{2T}-\frac{5}{2}\right)n_e\nabla T\right]f_{e0}=\left(C^{\ell}_{ee}+\nu_{ei}L\right)f'_{e1} \tag{50}$$

where C^{ℓ}_{ee} is the linearized Fokker-Planck operator, $L=\left(1/2\right)\left(\partial/\partial\vec{\upsilon}'\right)\cdot\left(\vec{\upsilon}'\vec{\upsilon}'-\upsilon'^2\vec{I}\right)\cdot\partial/\partial\vec{\upsilon}'$ is the pitch angle scattering operator, and $\nu_{ei}=4\pi Z_i^2e^4n_i\ln\Lambda/m_e\upsilon'^3$. In the approximate electron-ion collision operator, the term $C_{ei}\left(f_{e0},f'_{i1}\right)=\nu_{ei}\left(m_e\upsilon'_{\|}\,u_{\|i}/T\right)f_{e0}$ has been neglected in view of the estimate for $u_{\|i}$.

Using the approximation $C_{ie}\left(f'_{i1}, f_{e0}\right) = 0,\ C_{ie}\left(f_{i0}, f'_{e1}\right) = \left(\upsilon'_{\|} F_{\| ei} / p_i\right) f_{i0}$ where $F_{\| ei} = \int d^3\upsilon' m_e \upsilon'_{\|} \nu_{ei} L f'_{e1}$, and the relation $\vec{b} \cdot \left(\nabla p_e + n_e e \vec{E}^{(0)}\right) = F_{\| ei}$ that follows from the electron equation, the ion equation simplifies to

$$\vec{\upsilon}'_{\|} \cdot \left(\frac{m_i \upsilon'^2}{2T} - \frac{5}{2}\right) \frac{\nabla T}{T} f_{i0} = C^{\ell}_{ii} f'_{i1} \quad . \tag{51}$$

From the solution to Eq. (50), the electron parallel velocity and heat flux can be expressed in the form

$$u_{\| e} = -\frac{\tau_{ei}}{m_e n_e} \left[\lambda_{11} \left(\nabla_{\|} p_e + n_e e E^{(0)}_{\|}\right) + \lambda_{12} n_e \nabla_{\|} T\right] \tag{52}$$

$$q_{\| e} = \frac{5}{2} p_e u_{\| e} - \frac{T \tau_{ei}}{m_e} \left[\lambda_{21} \left(\nabla_{\|} p_e + n_e e E^{(0)}_{\|}\right) + \lambda_{22} n_e \nabla_{\|} T\right] \tag{53}$$

where $\tau_{ei} = 3\sqrt{m_e} T^{3/2} / 4\sqrt{2\pi} Z_i^2 e^4 n_i \ln \Lambda$. The Z_i dependent dimensionless coefficients $\lambda_{11}, \lambda_{12} \left(= \lambda_{21}\right), \lambda_{22}$ are introduced in Ref. 8, which also provides tabulated values for them. In particular, for $Z_i = 1$, $\lambda_{11} = 1.975, \lambda_{12} = \lambda_{21} = 1.389, \lambda_{22} = 4.174$. The presence of the term involving $u_{\| e}$ in Eq. (53) in comparison with a similar equation in Ref. 8 is because the heat flux in that reference is relative to the electron velocity rather than the mass velocity. For completeness, the solution of Eq. (50) based on a three-term expansion of the distribution function in the Sonine polynomials is described in Appendix B. Sample results of the coefficients are also given there.

The parallel electric field is solved for from Eq. (52) after replacing $u_{\| e}$ by the parallel current using $j_{\|} = -n_e e u_{\| e}$, which is accurate in two orders of $\sqrt{m_e / m_i}$. The result is

$$E^{(0)}_{\|} = -\frac{\nabla_{\|} p_e}{n_e e} - \frac{\lambda_{12}}{\lambda_{11}} \frac{\nabla_{\|} T}{e} + \frac{j_{\|}}{\sigma_{\|}} \tag{54}$$

where $\sigma_{\|} = \lambda_{11} n_e e^2 \tau_{ei} / m_e$. Using the above equation to eliminate $E^{(0)}_{\|}$, Eq. (53) becomes

$$q_{\| e} = -\left(\lambda_{22} - \frac{\lambda_{12}}{\lambda_{11}} \lambda_{21}\right) \frac{p_e \tau_{ei}}{m_e} \nabla_{\|} T - \left(\frac{\lambda_{12}}{\lambda_{11}} + \frac{5}{2}\right) \frac{T}{e} j_{\|} \quad . \tag{55}$$

The explicit dependence on current density for the heat flux differs from most existing works. From solving Eq. (51), the ion parallel heat flux is given by

$$q_{\parallel i} = -3.91 \frac{p_i \tau_i}{m_i} \nabla_{\parallel} T \tag{56}$$

where $\tau_i = 3\sqrt{m_i} T^{3/2} \big/ 4\sqrt{\pi} Z_i^4 e^4 n_i \ln \Lambda$.

B. Perpendicular velocities and heat fluxes

For the solution of Eq. (37), which gives the perpendicular components of velocities, heat fluxes, and electric field, the assumption is made that the ratio of collision frequency to gyro-frequency is small for both electron and ion, which is the case for strongly magnetized plasmas. This represents a supplementary expansion within the master expansion in δ , and allows closed forms for the fluxes to be obtained for plasmas with an arbitrary number of ion species. As described in Ref. 8, the leading order solution $f_{a1}'^{(0)}$ of Eq. (37) is found by neglecting the collision terms. The result is

$$f_{a1}'^{(0)} = \frac{m_a \vec{\upsilon}'}{T} \cdot \hat{b} \times \frac{c}{n_a e_a B} \left[\vec{A}_{\perp a} - n_a e_a \vec{E}_{\perp}'^{(0)} + \left(\frac{m_a \upsilon'^2}{T} - \frac{5}{2} \right) n_a \nabla_{\perp} T \right] f_{a0} \tag{57}$$

where

$$\vec{A}_{\perp a} = \nabla_{\perp} p_a - \frac{\rho_a}{\rho} \left(\nabla_{\perp} p - \frac{\vec{j}^{(1)}}{c} \times \vec{B} \right) \quad . \tag{58}$$

The leading order fluxes are determined using Eq. (57). Correction to the fluxes in the next order (in the ratio of collision frequency over gyrofrequency) is found by first evaluating the collision terms using $f_{a1}'^{(0)}$. The collision terms consist of the friction force $\vec{F}_{ab} = \int d^3\upsilon' m_a \vec{\upsilon}' \left[C_{ab}\left(f_{a1}'^{(0)}, f_{b0} \right) + C_{ab}\left(f_{a0}, f_{b1}'^{(0)} \right) \right]$ and heat friction $\vec{G}_{ab} = \int d^3\upsilon' m_a \vec{\upsilon}' \left(m_a \upsilon'^2 / 2T - 5/2 \right) \left[C_{ab}\left(f_{a1}'^{(0)}, f_{b0} \right) + C_{ab}\left(f_{a0}, f_{b1}'^{(0)} \right) \right]$.

They are given by

$$\vec{F}_{ab} = \frac{1}{\sqrt{1 + m_a/m_b}} \frac{m_a n_a}{\tau_{ab}} \frac{1}{eB} \hat{b} \times \left[\frac{\vec{A}_{\perp b}}{Z_b n_b} - \frac{\vec{A}_{\perp a}}{Z_a n_a} + \frac{3}{2} \frac{1}{m_a + m_b} \left(\frac{m_b}{Z_a} - \frac{m_a}{Z_b} \right) \nabla T \right] \tag{59}$$

$$\vec{G}_{ab} = \frac{1}{(1 + m_a/m_a)^{3/2}} \frac{m_a n_a}{\tau_{ab}} \frac{1}{eB} \hat{b} \times \left\{ \frac{3}{2} \left(\frac{\vec{A}_{\perp a}}{Z_a n_a} - \frac{\vec{A}_{\perp b}}{Z_b n_b} \right) - \frac{1}{1 + m_a/m_a} \left[\frac{1}{Z_a} \left(\frac{13}{4} + 4\frac{m_a}{m_b} + \frac{15}{2} \frac{m_a^2}{m_b^2} \right) - \frac{1}{Z_b} \frac{27}{4} \frac{m_a}{m_b} \right] \nabla T \right\} \tag{60}$$

where $\tau_{ab} = 3\sqrt{m_a}T^{3/2} \big/ 4\sqrt{2\pi}Z_a^2 Z_b^2 e^4 n_b \ln\Lambda$. Performing the operations $\int d^3 v' m_a \vec{v}'$ and $\int d^3 v' m_a \vec{v}' \left(m_a v'^2 / T - 5/2 \right)$ on Eq. (37), the perpendicular velocities and heat fluxes accurate to two orders in the ratio of gyro-radius over scale length are found to be

$$n_a \vec{u}_{\perp a} = \frac{c}{e_a B} \hat{b} \times \left(\vec{A}_a - n_a e_a E_{\perp}'^{(0)} - \sum_b \vec{F}_{ab} \right) \tag{61}$$

$$\vec{q}_{\perp a} = \frac{5}{2} p_a \vec{u}_{\perp a} + \frac{cT}{e_a B} \hat{b} \times \left(\frac{5}{2} n_a \nabla T - \sum_b \vec{G}_{ab} \right) \tag{62}$$

where the terms that depend on the collision integrals are formally smaller. The condition $\sum_a \rho_a \vec{u}_{\perp a} = 0$ that follows from the perpendicular component of Eq. (41) determines $\vec{E}_{\perp}'^{(0)}$ in the form

$$\vec{E}_{\perp}'^{(0)} = \frac{1}{\rho} \sum_a \frac{m_a}{e_a} \left(\vec{A}_{\perp a} - \sum_b \vec{F}_{ab} \right) \quad . \tag{63}$$

Specializing to a simple plasma and taking in addition the parameter $\sqrt{m_e/m_i}$ to be small, we have the estimates $u_{\perp e} \sim u_{\perp i} \sim q_{\perp e}/p_e \sim q_{\perp i}/p_i \sim \left(\rho_{Be}/\ell \right) \bar{v}_e \sim \left(\rho_{Bi}/\ell \right) \bar{v}_i$ to leading order, where ρ_{Be}/ℓ and ρ_{Bi}/ℓ are ratios of gyro-radius to perpendicular scale length for electron and ion respectively. Corrections are smaller by the order of the ratio ρ_{Be}/ℓ for $u_{\perp e}, u_{\perp i}, q_{\perp e}$, and the ratio ρ_{Bi}/ℓ for $q_{\perp i}$. In this case, $\vec{A}_{\perp e} = \nabla_{\perp} p_e, \vec{A}_{\perp i} = -\nabla_{\perp} p_e + \vec{j}^{(1)} \times \vec{B} / c$, $\vec{u}_{\perp i} = 0$, $\vec{j}_{\perp} = -n_e e \vec{u}_{\perp e}$ and Eq. (63) reduces to

$$\vec{E}_{\perp}'^{(0)} = \frac{1}{n_e e} \left(-\nabla_{\perp} p_e + \frac{\vec{j}^{(1)}}{c} \times \vec{B} \right) + \frac{\vec{j}_{\perp}^{(1)}}{\sigma_{\perp}} - \frac{3}{2} \frac{1}{\Omega_e \tau_{ei}} \frac{1}{Z_i e} \hat{b} \times \nabla T \tag{64}$$

with $\sigma_{\perp} = n_e e^2 \tau_{ei} / m_e$ and $\Omega_e = eB/m_e c$, in which the last two terms are formally smaller.

Combining Eqs. (33), (54) and (64), and removing superscripts, the relation

$$\vec{E} + \frac{\vec{V}}{c} \times \vec{B} = -\frac{\nabla p_e}{n_e e} - \frac{\lambda_{12}}{\lambda_{11}} \frac{\nabla_{\parallel} T}{e} + \frac{j_{\parallel}}{\sigma_{\parallel}} + \frac{\vec{j} \times \vec{B}}{n_e e c} + \frac{\vec{j}_{\perp}}{\sigma_{\perp}} - \frac{3}{2} \frac{1}{\Omega_e \tau_{ei}} \frac{1}{Z_i e} \hat{b} \times \nabla T \tag{65}$$

follows, which can be regarded as a generalized Ohm's law. It can be used for eliminating the electric field in the fluid equations valid through the first order frequency scale $\delta\omega_0$ (and second order in the ratio of collision frequency over gyrofrequency).

Using Eqs. (60) and (62), the electron and ion perpendicular heat fluxes are computed to be

$$\vec{q}_{\perp e} = -\frac{5}{2}\frac{cp_e}{eB}\hat{b}\times\nabla T - \frac{5}{2}\frac{T}{e}\vec{j}_{\perp} - \frac{n_e\rho_{Be}^2}{\sqrt{2}\tau_{ee}}\left(1+\frac{13Z_i}{4\sqrt{2}}\right)\nabla_{\perp}T + \frac{3}{2}\frac{1}{\Omega_e\tau_{ei}}\frac{T}{e}\vec{j}\times\hat{b} \tag{66}$$

$$\vec{q}_{\perp i} = \frac{5}{2}\frac{cp_i}{Z_ieB}\hat{b}\times\nabla T - \frac{n_i\rho_{Bi}^2}{\tau_i}\left(1+\frac{15}{2\sqrt{2}}\sqrt{\frac{m_e}{m_i}}\frac{1}{Z_i}\right)\nabla_{\perp}T \tag{67}$$

where the gyro-radii are defined by $\rho_{Be} = c\sqrt{2m_eT}\big/eB,\ \rho_{Bi} = c\sqrt{2m_iT}\big/Z_ieB$. In these expressions, the first two terms of Eq. (66) and the first term of Eq. (67) are formally larger than the rest in an expansion in gyro-radius over gradient length.

C. Parallel viscous stress tensors

The part of the stress tensor that depends on the gyro-phase averaged distribution functions is derived from the solution of Eq. (36), and can be referred to as the parallel viscous stress. The electron and ion contributions to the parallel viscous stress have the estimates $\pi_i \sim (\lambda/\ell)p,\quad \pi_e \sim \sqrt{m_e/m_i}\pi_i$. In the electron version of Eq. (36), the electron ion collision can be approximated by pitch-angle scattering. Solution by expansion in Sonine polynomials as shown in Appendix B leads to the electron contribution:

$$\vec{\vec{\pi}}_{\parallel e} = -\alpha_e p_e \tau_{ee}\vec{\vec{W}}_0 \tag{68}$$

where

$$\vec{\vec{W}}_0 = \frac{3}{2}\left(\hat{b}\hat{b} - \frac{1}{3}\vec{\vec{I}}\right)\hat{b}\cdot\vec{\vec{W}}\cdot\hat{b} \tag{69}$$

$$\vec{\vec{W}} = \nabla\vec{V}^{(0)} + \left(\nabla\vec{V}^{(0)}\right)^T - \frac{2\vec{\vec{I}}}{3}\nabla\cdot\vec{V}^{(0)} \tag{70}$$

and α_e is a numerical coefficient that depends on Z_i, being equal to 0.73 for $Z_i = 1$.

To determine the ion contribution with the same degree of accuracy in an expansion in $\sqrt{m_e/m_i}$, it is necessary to keep the ion-electron collision operator in Eq. (36) to leading order, which is neglected in Ref. 4. This operator is

$$C_{ie}\left(f'_{i1}, f_{e0}\right) = \frac{1}{\tau_{ie}}\sqrt{\frac{m_e}{m_i}}\frac{\partial}{\partial\vec{v}'}\cdot\left(\vec{v}'f'_{i1} + \frac{T}{m_i}\frac{\partial f'_{i1}}{\partial\vec{v}'}\right) \tag{71}$$

where $\tau_{ie} = 3\sqrt{m_i}T^{3/2} / 4\sqrt{2\pi}Z_i^2 n_e e^4 \ell n\Lambda$. Again using expansion in Sonine polynomials as shown in Appendix B, and treating C_{ie} as a perturbation, the ion contribution is found to be

$$\vec{\vec{\pi}}_{\|i} = -\left(0.96 - 3.17\frac{1}{Z_i}\sqrt{\frac{m_e}{m_i}}\right)p_i\tau_i\vec{\vec{W}}_0 \quad . \tag{72}$$

The correction terms of the order of $\sqrt{m_e/m_i}$ in both Eqs. (67) and (72) are not included in Ref. 4. They have the same order of magnitude as some of the terms retained in Eq. (66) for the electron heat flux and Eq. (68) for the electron parallel viscosity respectively.

D. Gyro-viscous and Perpendicular Stress Tensors

The parts of the stress tensor that depend on the gyro-phase dependent parts of the distribution functions obeying Eq. (38) have the estimates $\pi_i \sim \left(\rho_{Bi}/\ell\right)p, \quad \pi_e \sim \left(m_e/m_i\right)\pi_i$, with corrections smaller by the factors ρ_{Bi}/ℓ and ρ_{Be}/ℓ, respectively. Just as for the perpendicular velocity and heat flux, these components can be calculated for a multi-component plasma. First, writing the magnetic field term in Eq. (38) as $-\Omega_a^{-1}\,\partial f'_{a1}/\partial\theta$ where $\Omega_a = e_a B/m_a c$ and θ is the gyro-phase angle, the leading order distribution functions are found by neglecting the collision term and integrating over θ, resulting in the equation

$$f'^{(0)}_{a1} = \frac{1}{\Omega_a}\frac{m_a}{2T}\int\left(\vec{v}'\vec{v}' - \left\langle\vec{v}'\vec{v}'\right\rangle\right)d\theta : \vec{\vec{W}} f_{a0} \tag{73}$$

from which the gyro-viscous stress can be found by direct evaluation to be

$$\vec{\vec{\pi}}_{\wedge a} = \frac{p_a}{2\Omega_a}\left(\vec{\vec{W}}_3 + 2\vec{\vec{W}}_4\right) \quad . \tag{74}$$

In this equation, the components of the strain rate tensor are defined using the projection tensors $\vec{\vec{P}}_{\|} = \hat{b}\hat{b}, \vec{\vec{P}}_{\perp} = \vec{\vec{I}} - \hat{b}\hat{b}$, $\vec{\vec{P}}_{\wedge}$ with components $\left(\vec{\vec{P}}_{\wedge}\right)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}b_\gamma$ as follows:

$$\vec{\vec{W}}_3 = \frac{1}{2}\left(\vec{\vec{P}}_{\perp}\vec{\vec{W}}\vec{\vec{P}}_{\wedge} - \vec{\vec{P}}_{\wedge}\vec{\vec{W}}\vec{\vec{P}}_{\perp}\right) \tag{75}$$

$$\vec{\vec{W}}_4 = \vec{\vec{P}}_{\|}\vec{\vec{W}}\vec{\vec{P}}_{\wedge} - \vec{\vec{P}}_{\wedge}\vec{\vec{W}}\vec{\vec{P}}_{\|} \quad . \tag{76}$$

In the next order in ρ_{Ba}/ℓ , Eq. (38) becomes

$$-\frac{1}{\Omega_a}\frac{\partial f'^{(1)}_{a1}}{\partial\theta} = C_{ab}\left(f'^{(0)}_{a1}, f_{b0}\right) + C_{ab}\left(f_{a0}, f'^{(0)}_{b1}\right) \quad . \tag{77}$$

Upon multiplication by $\int\left(\vec{v}'\vec{v}' - \left\langle\vec{v}'\vec{v}'\right\rangle\right)d\theta$ on both sides and integrating over velocities, the contribution of species a to the viscous stress tensor in this order, which can be called the perpendicular stress, can be expressed in the form

$$\vec{\vec{\pi}}_{\perp a} = \sum_b \vec{\vec{\pi}}_{\perp ab} \tag{78}$$

where, from evaluation of the integrals involving the collision terms, it is found that

$$\vec{\vec{\pi}}_{\perp ab} = -\frac{1}{\left(1 + m_a/m_b\right)^{3/2}}\left[\left(\frac{3}{10} + \frac{1}{2}\frac{m_a}{m_b}\right)\frac{1}{\Omega_a} - \frac{1}{5}\frac{m_a}{m_b}\frac{1}{\Omega_b}\right]\frac{p_a}{\Omega_a \tau_{ab}}\left(\vec{\vec{W}}_1 + 4\vec{\vec{W}}_2\right) \tag{79}$$

with

$$\vec{\vec{W}}_1 = \vec{\vec{P}}_\perp \vec{\vec{W}} \vec{\vec{P}}_\perp - \left(tr \vec{\vec{P}}_\perp \vec{\vec{W}} \vec{\vec{P}}_\perp / 2\right)\vec{\vec{P}}_\perp \tag{80}$$

$$\vec{\vec{W}}_2 = \vec{\vec{P}}_\parallel \vec{\vec{W}} \vec{\vec{P}}_\perp + \vec{\vec{P}}_\perp \vec{\vec{W}} \vec{\vec{P}}_\parallel \tag{81}$$

The contribution $\vec{\vec{\pi}}_{\perp ab}$ is negligible when a refers to an ion species and b to the electrons.

Combining electron and ion contributions, the full stress tensor for a simple plasma is given by

$$\vec{\vec{\pi}} = -\left[\left(0.96 - 3.17\sqrt{\frac{m_e}{m_i}}\right)p_i\tau_i + \alpha_e p_e \tau_{ee}\right]\vec{\vec{W}}_0 + \frac{p_i}{2\Omega_i}\left(\vec{\vec{W}}_3 + 2\vec{\vec{W}}_4\right) - \frac{3}{10}\frac{p_i}{\Omega_i^2 \tau_i}\left(\vec{\vec{W}}_1 + 4\vec{\vec{W}}_2\right) \quad . \tag{82}$$

Finally, direct calculation shows that the entropy production rate θ is given by

$$\begin{aligned} T\theta = {} & \frac{j_\parallel^2}{\sigma_\parallel} + \frac{j_\perp^2}{\sigma_\perp} + \left[\left(\lambda_{22} - \frac{\lambda_{12}}{\lambda_{11}}\lambda_{21}\right)\frac{p_e \tau_{ei}}{m_e} + 3.91\frac{p_i \tau_i}{m_i}\right]\left(\nabla_\parallel \ell n T\right)^2 \\ & + \left[\frac{n_e \rho_{Be}^2}{\sqrt{2}\tau_{ee}}\left(1 + \frac{13 Z_i}{4\sqrt{2}}\right) + \frac{n_i \rho_{Bi}^2}{\tau_i}\left(1 + \frac{15}{2\sqrt{2}}\sqrt{\frac{m_e}{m_i}}\frac{1}{Z_i}\right)\right]\left(\nabla_\perp \ell n T\right)^2 \quad . \\ & + \left[\left(0.48 + 2.24\sqrt{\frac{m_e}{m_i}}\right)p_i \tau_i + 0.37 p_e \tau_{ee}\right]\vec{\vec{W}}_0 : \vec{\vec{W}}_0 + \frac{3}{20}\frac{p_i}{\Omega_i^2 \tau_i}\left(\vec{\vec{W}}_1 : \vec{\vec{W}}_1 + 4\vec{\vec{W}}_2 : \vec{\vec{W}}_2\right) \end{aligned} \tag{83}$$

Braginskii[4] obtains the momentum and temperature equations for the electron and ion fluids for a simple plasma. When the electron and ion temperatures are set equal to each other, the results can be compared with those in this section. In making the comparison, a few steps must be taken to manage the difference in forms of Braginskii's equations with the ones in this work. One step is to replace the ion-electron velocity difference $\vec{u} = \vec{V}_i - \vec{V}_e$ by the current density using $\vec{u} = \vec{j}/n_e e$ everywhere it occurs. Another is to identify the ion velocity with the mass velocity, and express the electron velocity in terms of it and the current density. One more is to recognize that the electron heat flux in Braginskii's equations is defined relative to the electron fluid velocity instead of the mass velocity. It then follows that the present work agrees with Braginskii's, except for the terms in Eqs. (67) and (72) that include corrections of the order of $\sqrt{m_e/m_i}$, which are of the same order of magnitude as the corresponding quantities for the electron fluid, and the neglect of the gyro-viscous and perpendicular stress tensors for the latter, which would represent corrections of order m_e/m_i to the contributions from the ion fluid (the retention of contributions to the gyro-viscous and perpendicular stress tensors from electrons cannot be justified when the ion contributions are correct only to the order $\sqrt{m_e/m_i}$).

Equation (65) can also be found in Braginskii's article, where it is also referred to as the generalized Ohm's law. However, it is derived there from the electron momentum equation by neglecting the acceleration and the viscosity terms. Ref. 9 regards the generalized Ohm's law to arise from the requirement of an equation for $\partial\vec{j}/\partial t$, which is equivalent to the electron momentum equation when the approximation $m_e/m_i << 1$ is taken. Following this line of reasoning, the viscosity term can occur in the generalized Ohm's law, as is found, for example, in Ref. 10. This is not true in the present approach, in which the generalized Ohm's law is an expression for the electric field in terms of fluid variables including the current density, and viscosity does not influence the electric field in the frequency scale $\delta\omega_0$.

VI. Plasma with Two Ion Species

Certain general features in the formulation in Sections II, III and IV, such as the inter-diffusions of ions, are not present for simple plasmas. In this section, plasmas with two ion species, denoted by i and j, are considered. Examples of applications are to plasmas with one main ion species and an impurity species, and plasmas with deuterium and tritium ions. Electron mass is taken to be small in comparison with ion masses.

Consider first the calculation of the parallel velocities and heat flux from Eq. (35). For each ion species, collisions with electrons can be approximated in the same way as for simple plasmas in the previous section. Thus, for $a = i$ and j, $C_{ae}\left(f'_{a1}, f_{e0}\right) = 0$, $C_{ae}\left(f_{a0}, f'_{e1}\right) = \left(\upsilon'_\parallel F_{\parallel ea}/p_a\right) f_{a0}$ where $F_{\parallel ea} = \int d^3\upsilon' m_e \upsilon'_\parallel \nu_{ea} L f'_{e1}$. The parallel frictions $F_{\parallel ei}$ and $F_{\parallel ej}$ can be eliminated with the help of the parallel force balance equation $\nabla_\parallel p_e + n_e e E_\parallel^{(0)} = F_{\parallel ei} + F_{\parallel ej}$ that follows from the electron version of Eq. (35), and the relation $F_{\parallel ei} : F_{\parallel ej} = Z_i^2 n_i : Z_j^2 n_j$. This gives rise to the pair of ion equations decoupled from the electron equation:

$$\frac{\upsilon'_\parallel}{p_i}\left[D_{ij} + \left(\frac{m_i \upsilon'^2}{2T} - \frac{5}{2}\right) n_i \nabla_\parallel T\right] f_{i0} = C_{ii}\left(f'_{i1}, f_{i0}\right) + C_{ii}\left(f_{i0}, f'_{i1}\right) + C_{ij}\left(f'_{i1}, f_{j0}\right) + C_{ij}\left(f_{i0}, f'_{j1}\right) \qquad (84)$$

$$\frac{\upsilon'_\parallel}{p_j}\left[-D_{ij}+\left(\frac{m_j\upsilon'^2}{2T}-\frac{5}{2}\right)n_j\nabla_\parallel T\right]f_{j0}=C_{jj}\left(f'_{j1},f_{j0}\right)+C_{jj}\left(f_{j0},f'_{j1}\right)+C_{ji}\left(f'_{j1},f_{i0}\right)+C_{ji}\left(f_{j0},f'_{i1}\right) \qquad (85)$$

where

$$D_{ij}=\frac{\rho_i\rho_j}{\rho_{ij}}\left(\frac{\nabla_\parallel p_i}{\rho_i}-\frac{\nabla_\parallel p_j}{\rho_j}\right)-Z_in_ieE_\parallel^{(0)}-\frac{\rho_i}{\rho_{ij}}\nabla_\parallel p_e+\frac{Z_i^2n_i}{Z_i^2n_i+Z_j^2n_j}\left(\nabla_\parallel p_e+n_eeE_\parallel^{(0)}\right) \quad . \qquad (86)$$

Also, $n_{ij}=n_i+n_j$, $p_{ij}=p_i+p_j$ and $\rho_{ij}=\rho_i+\rho_j$. The solution of these equations yields the difference in parallel ion velocities and the total ion parallel heat flux, which can be expressed in the form

$$u_{\parallel i}-u_{\parallel j}=-\frac{1}{\sqrt{m_im_j}}\frac{\overline{\tau}_{ij}}{n_{ij}}\left(l_{11}D_{ij}+l_{12}n_{ij}\nabla_\parallel T\right) \qquad (87)$$

$$q_{\parallel i}+q_{\parallel j}=\frac{5}{2}\left(p_iu_{\parallel i}+p_ju_{\parallel j}\right)-\frac{T}{\sqrt{m_im_j}}\overline{\tau}_{ij}\left(l_{21}D_{ij}+l_{22}n_{ij}\nabla_\parallel T\right) \qquad (88)$$

where $\overline{\tau}_{ij}=3\left(m_im_j\right)^{1/4}T^{3/2}\Big/4\sqrt{2\pi}Z_i^2Z_j^2n_{ij}e^4\ell n\Lambda$, and $l_{11},l_{12}(=l_{21}),l_{22}$ are dimensionless coefficients depending on m_i/m_j, Z_i/Z_j and n_i/n_j. Invoking the condition $\rho_iu_{\parallel i}+\rho_ju_{\parallel j}=0$, which follows from the parallel component of Eq. (41) with neglect of the electron term, the deviation from the common parallel mass velocity for individual ions can be found from

$$u_{\parallel i}=\frac{\rho_j}{\rho_{ij}}\left(u_{\parallel i}-u_{\parallel j}\right) \qquad u_{\parallel j}=-\frac{\rho_i}{\rho_{ij}}\left(u_{\parallel i}-u_{\parallel j}\right) \qquad (89)$$

In Appendix B, the solution of Eqs. (84) and (85) by expansion in the Sonine polynomials is described and some results for the coefficients l_{mn} are presented.

Equations (87) and (88) involve the parallel electric field $E_\parallel^{(0)}$ through D_{ij}, which can be determined by considering the electron version of Eq. (35). Here the same approximation for the electron-ion collision operator applies for i and j. The term involving $u_{\parallel i}$ or $u_{\parallel j}$ in this approximation is of a higher order in $\sqrt{m_e/m_i}$ or $\sqrt{m_e/m_j}$. It is retained so that the electron parallel heat flux can be computed with a correction of this order, which is comparable to the ion parallel heat flux. Defining

$$\overline{u}_{ij}=\frac{Z_i^2n_iu_{\parallel i}+Z_j^2n_ju_{\parallel j}}{Z_i^2n_i+Z_j^2n_j}=\left(\frac{Z_i^2}{m_i}-\frac{Z_j^2}{m_j}\right)\frac{\rho_i\rho_j}{\rho_{ij}}\frac{u_{\parallel i}-u_{\parallel j}}{Z_i^2n_i+Z_j^2n_j} \qquad (90)$$

the shifted distribution function $f''_{e1} = f'_{e1} - \left(m_e v'_\parallel \bar{u}_{ij} / T \right) f_{e0}$ satisfies Eq. (50) with ν_{ei} replaced by $\nu_{eij} = 4\pi \left(Z_i^2 n_i + Z_j^2 n_j \right) e^4 \ln\Lambda / m_e v'^3$. Consequently, Eqs. (52) and (53) still hold, with $u_{\parallel e}$ replaced by $\int d^3v' v'_\parallel f''_{e1} / n_e$ and τ_{ei} replaced by τ_{eij}, which is obtained from τ_{ei} by replacing Z_i with $Z_{eff} = \left(Z_i^2 n_i + Z_j^2 n_j \right) / n_e$. Introducing the ion parallel current

$$j_{\parallel ij} = e\left(Z_i n_i u_{\parallel i} + Z_j n_j u_{\parallel j} \right) = e\left(\frac{Z_i}{m_i} - \frac{Z_j}{m_j} \right) \frac{\rho_i \rho_j}{\rho_{ij}} \left(u_{\parallel i} - u_{\parallel j} \right) \quad , \tag{91}$$

the parallel electric field $E_\parallel^{(0)}$ can be solved for from the modified form of Eq. (52) and expressed in the form

$$E_\parallel^{(0)} = -\frac{\nabla_\parallel p_e}{n_e e} - \frac{\lambda_{12}}{\lambda_{11}} \frac{\nabla_\parallel T}{e} + \frac{1}{\sigma_\parallel} \left(j_\parallel - j_{\parallel ij} + n_e e \bar{u}_{ij} \right) \tag{92}$$

Together with Eq. (87) and the definitions in Eqs. (90) and (91), this relation allows the simultaneous determination of $E_\parallel^{(0)}$ and $u_{\parallel i} - u_{\parallel j}$ in terms of the independent fluid variables.

Eliminating $E_\parallel^{(0)}$ from the expression for $q_{\parallel e}$ given by the modified form of Eq. (53), the electron parallel heat flux is

$$q_{\parallel e} = -\left(\lambda_{22} - \frac{\lambda_{12}}{\lambda_{11}} \lambda_{21} \right) \frac{p_e \tau_{eij}}{m_e} \nabla_\parallel T - \left(\frac{\lambda_{12}}{\lambda_{11}} + \frac{5}{2} \right) \frac{T}{e} \left(j_\parallel - j_{\parallel ij} \right) - \frac{\lambda_{12}}{\lambda_{11}} p_e \bar{u}_{ij} \quad . \tag{93}$$

In this expression, the terms involving $j_{\parallel ij}$ and $\bar{u}_{ij}$ are corrections in the expansion in $\sqrt{m_e/m_i}$, but are of the same order of magnitude as the ion parallel heat flux given by Eq. (88).

The perpendicular velocities and heat fluxes can be directly obtained from Eqs. (61) and (62) of the previous section. Treating the ratio of electron mass to ion masses as small, the relations

$$\vec{A}_{\perp e} = \nabla_\perp p_e \tag{94}$$

$$\vec{A}_{\perp i} = \frac{\rho_i \rho_j}{\rho_{ij}} \left(\frac{\nabla_\perp p_i}{\rho_i} - \frac{\nabla_\perp p_j}{\rho_j} \right) + \frac{\rho_i}{\rho_{ij}} \left(-\nabla_\perp p_e + \frac{\vec{j}}{c} \times \vec{B} \right) \tag{95}$$

$$\vec{A}_{\perp j} = -\frac{\rho_i \rho_j}{\rho_{ij}} \left(\frac{\nabla_\perp p_i}{\rho_i} - \frac{\nabla_\perp p_j}{\rho_j} \right) + \frac{\rho_j}{\rho_{ij}} \left(-\nabla_\perp p_e + \frac{\vec{j}}{c} \times \vec{B} \right) \tag{96}$$

can be used in these equations. Thus, the generalized forces on which the velocities and fluxes depend are $\nabla_\perp p_e$, $\nabla_\perp T, \vec{j}_\perp$, and $\nabla_\perp p_i/\rho_i - \nabla_\perp p_j/\rho_j$. In addition, the friction terms $\vec{F}_{ie}, \vec{F}_{je}, \vec{G}_{ie}, \vec{G}_{je}$ can be neglected, and so is the term involving the electron mass in Eq. (63) for the perpendicular electric field. The latter equation takes the form

$$\vec{E}_\perp'^{(0)} = \frac{1}{e\rho_{ij}} \left(\frac{m_i}{Z_i} \vec{A}_{\perp i} + \frac{m_j}{Z_j} \vec{A}_{\perp j} \right) - \frac{1}{e\rho_{ij}} \left(\frac{m_i}{Z_i} - \frac{m_j}{Z_j} \right) \vec{F}_{ij} \quad . \tag{97}$$

The perpendicular velocity difference between the two ion species is

$$\vec{u}_{\perp i} - \vec{u}_{\perp j} = \frac{c}{eB} \hat{b} \times \left[\frac{\vec{A}_{\perp i}}{Z_i n_i} - \frac{\vec{A}_{\perp j}}{Z_j n_j} - \left(\frac{1}{Z_i n_i} - \frac{1}{Z_j n_j} \right) \vec{F}_{ij} \right] \quad . \tag{98}$$

An expression for $\vec{E} + \vec{V} \times \vec{B}/c$ correct to the frequency scale $\delta\omega_0$ in terms of the independent fluid variables is obtained by adding Eqs. (92) and (97). In contrast with simple plasmas, this generalized Ohm's law cannot be obtained from the momentum equation of the electron fluid.

The electron contribution to the parallel viscous stress is given by Eq. (68), in which the coefficient α_e depends on Z_{eff} instead of Z_i. As shown in Appendix B, the ion contribution can be written in the form

$$\vec{\vec{\pi}}_{\parallel i} + \vec{\vec{\pi}}_{\parallel j} = -\left[\left(\alpha_i + \beta_i \frac{n_e}{Z_j^2 n_{ij}} \sqrt{\frac{m_e}{m_i}} + \gamma_i \frac{n_e}{Z_i^2 n_{ij}} \sqrt{\frac{m_e}{m_j}} \right) p_i + \left(\alpha_j + \beta_j \frac{n_e}{Z_j^2 n_{ij}} \sqrt{\frac{m_e}{m_i}} + \gamma_j \frac{n_e}{Z_i^2 n_{ij}} \sqrt{\frac{m_e}{m_j}} \right) p_j \right] \sqrt{2} \overline{\tau}_{ij} \vec{\vec{W}}_0 \tag{99}$$

with six dimensionless coefficients depending on the mass ratio of the two ions.

The electron contributions to the gyro-viscous and perpendicular stresses are negligible as for a simple plasma. The ion contributions are the sums $\vec{\vec{\pi}}_{\wedge i} + \vec{\vec{\pi}}_{\wedge j}$ and $\vec{\vec{\pi}}_{\perp i} + \vec{\vec{\pi}}_{\perp j}$, where the individual terms are defined in Eqs. (74), (78) and (79) of the previous section.

VII. Classical Transport

In classical transport,[8] the component of the plasma velocity perpendicular to the magnetic field is considered to be smaller than the thermal velocity by one order in the ratio of gyroradius over scale length. The parallel velocity is either ignored in a slab model of the magnetic field, or considered also to be first order small in the ratio of mean-free-path over scale length. To obtain the fluid equations and calculate the transport coefficients, the distribution function is accordingly expanded in δ. In the present approach, this limit can be recovered directly from the fluid equations of the preceding sections by taking $\vec{E}^{(-1)} = 0$ and $\vec{V}^{(0)} = 0$.

First, it is clear from Section III that there is no time variation in the frequency scale of ω_0 ($\partial/\partial t^{(0)} = 0$), so that Eqs. (30) and (31) describe MHD equilibrium. The equilibrium equations continue to hold in the frequency scale $\delta\omega_0$ because, examining Eq. (4), $\vec{\pi} = 0$ on account of $\vec{V}^{(0)} = 0$, and

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial \vec{V}^{(0)}}{\partial t^{(1)}} + \frac{\partial \vec{V}^{(1)}}{\partial t^{(0)}} = 0 \quad . \tag{100}$$

There is now no equation that directly describes the dynamical evolution of the mass velocity $\vec{V}$. Instead, it is necessary to equate the time derivative of the MHD equilibrium equations to zero so that they continue to hold true if initially they are. The resulting equation also determines the electric field, and can be used in lieu of Eq. (49) for its elimination from Faraday's law and other fluid equations.

For simplicity, this procedure is applied to a simple plasma in a slab model where $\vec{B} = B_z\hat{z}$, $V_\parallel = 0$, and spatial variation is only in the x-direction. The current density is $\vec{j} = j_y\vec{y}$ and the equations for the fields are

$$\frac{\partial B_z}{\partial x} = -\frac{4\pi}{c} j_y \tag{101}$$

$$\frac{\partial B_z}{\partial t} = -c\frac{\partial E_y}{\partial x} \quad . \tag{102}$$

The mass continuity equation [Eq. (2)] and the temperature equation [Eq. (6)] simplify to

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}\left(nV_x\right) = 0 \tag{103}$$

and

$$\frac{3}{2}\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(\frac{3}{2}pV_x\right) + p\frac{\partial V_x}{\partial x} + \frac{\partial q_x}{\partial x} = 0 \quad , \tag{104}$$

while the momentum equation [Eq. (4)] becomes

$$\frac{\partial}{\partial x}\left(p + \frac{B_z^2}{8\pi}\right) = 0 \quad . \tag{105}$$

The x-component of the generalized Ohm's law [Eq. (65)] is

$$E_x + \frac{1}{c}V_yB_z = \frac{1}{n_ie}\frac{\partial p_i}{\partial x} \quad . \tag{106}$$

It involves the quantities E_x and V_y that appear in no other equations, and thus plays no further role. The y-component can be solved for V_x to express it as the sum of a $E \times B$ velocity and a diffusion velocity:

$$V_x = V_E + V_D \tag{107}$$

where

$$V_E = \frac{cE_y}{B_z} \tag{108}$$

$$V_D = -\frac{\rho_{Be}^2}{2\tau_{ei}}\left(\frac{1}{p_e}\frac{\partial p}{\partial x} - \frac{3}{2}\frac{1}{Z_i T}\frac{\partial T}{\partial x}\right) \quad . \tag{109}$$

Combining Eqs. (66), (67) and using Eqs. (101) and (105), the x-component of the heat flux is

$$q_x = -\frac{n_i \rho_{Bi}^2}{\tau_i}\frac{\partial T}{\partial x} - \frac{n_e \rho_{Be}^2}{\sqrt{2}\tau_{ee}}\left(1 + \frac{13Z_i}{4\sqrt{2}}\right)\frac{\partial T}{\partial x} + \frac{3}{4}\frac{\rho_{Be}^2}{\tau_{ei}}\frac{\partial p}{\partial x} \quad . \tag{110}$$

Upon differentiating Eq. (105) with respect to time and using Eqs. (102) and (104), the resulting equation can be cast in the form

$$\frac{\partial}{\partial x}\left[\left(\frac{5}{3}p + \frac{B_z^2}{4\pi}\right)\frac{\partial V_E}{\partial x} + \frac{5}{3}p\frac{\partial V_D}{\partial x} + V_D\frac{\partial p}{\partial x} + \frac{2}{3}\frac{\partial q_x}{\partial x}\right] = 0 \quad , \tag{111}$$

which is an equation for V_E, and hence E_y, allowing it to be eliminated from Eq. (102). We thus obtain a dynamically consistent system comprising Eqs. (102–104) and Eqs. (107–111), with Eq. (105) serving as an initial condition.

In so far as the solution of Eq. (111) depends on boundary conditions, the particle flux is nonlocal. For low values of β, defined as $8\pi p / B_z^2$, it becomes diffusive because of the estimate $V_E / V_D \sim \beta$ that follows from Eq. (111). This dependence of the nature of the particle flux on β is not apparent from existing works as the need to produce a dynamically consistent system of equations for the plasma and the fields is often overlooked.

VIII. Concluding Remarks

This work begins with the observation that the perpendicular mass velocity is not an independent fluid variable established by the relaxation of plasma in the time scale of gyromotion and collisions. And yet the perpendicular momentum equation is included in existing works. We find a resolution of the discrepancy by taking into consideration the need to couple the plasma equations to Ampere and Faraday's laws for the electromagnetic field. A central role is played by the generalized Ohm's law, which is used to eliminate the electric field. The resulting system of equations gives a consistent description of the dynamical evolution of the plasma and the fields in terms of fluid variables consisting of the species densities, common temperature, mass velocity, and magnetic field. The system reduces to the ideal magnetohydrodynamic equations in the zeroth order frequency scale, and describes transport processes in the first order. Existing results for simple plasmas are reproduced, and recipe for calculating transport coefficients for plasmas with two ion species are obtained. The limit of small mass velocity yields equations ascribable to classical transport, but the need to eliminate the electric field leads to a nonlocal nature for the process at finite beta. Although strictly speaking the equations are applicable only to plasmas with short collisional mean free path, features emphasized in the present work such as the need for dynamic consistency for the joint evolution of plasma and field variables and the role and form of the generalized Ohm's law, might still prove useful in modeling high temperature plasmas.

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APPENDIX A: JUSTIFICATION FOR THE NEGLECT OF DISPLACEMENT CURRENT AND QUASI-NEUTRALITY

Using the orderings $cE_\perp / B\overline{\upsilon} \sim \delta^0$ and the fact that the most rapid time variation considered corresponds to the frequency scale ω_0, the estimate

$$\frac{\partial E_\perp / \partial t}{c\nabla \times B} \sim \frac{\omega_0 E_\perp \ell}{cB} \sim \left(\frac{\overline{\upsilon}}{c}\right)^2 \tag{A1}$$

is obtained. The similar ratio with $E_\parallel$ replacing $E_\perp$ is smaller by a factor δ, as $E_\parallel / E_\perp \sim \delta$. Thus for nonrelativistic motions the displacement current can be neglected.

In the equation of charge conservation, the time rate of change of the charge density $\rho_e = \nabla \cdot E / 4\pi$ has the estimate

$$\frac{\partial \rho_e / \partial t}{\nabla \cdot \vec{j}} \sim \frac{\omega_0 E_\perp \ell}{cB} \sim \left(\frac{\overline{\upsilon}}{c}\right)^2 \tag{A2}$$

using $E_\perp$ and Ampere's law. On the other hand, using $E_\parallel$ with its estimate $eE_\parallel \ell / T \sim \delta^0$ and $j \sim \delta n e \overline{\upsilon}$,

$$\frac{\partial \rho_e / \partial t}{\nabla \cdot \vec{j}} \sim \frac{\omega_0 E_\parallel}{4\pi \delta n e \overline{\upsilon}} \sim \delta^{-1}\left(\frac{\lambda_D}{\ell}\right)^2 \tag{A3}$$

where $\lambda_D = \left(T / 4\pi n e^2\right)^{1/2}$ is the Debye length.

In the momentum equation, the force density due to the electric field has the estimate

$$\frac{\rho_e E_\perp}{\rho \partial V / \partial t} \sim \frac{E_\perp^2 / \ell}{4\pi \rho \omega_0 \overline{\upsilon}} \sim \left(\frac{\upsilon_A}{c}\right)^2 \tag{A4}$$

using $E_\perp$ where $\upsilon_A = \left(B^2 / 4\pi\rho\right)^{1/2}$ is the Alfven velocity, and

$$\frac{\rho_e E_\parallel}{\rho \partial V / \partial t} \sim \frac{\rho_e T / e\ell}{\rho \omega_0 \overline{\upsilon}} \sim \frac{\rho_e}{ne} \sim \left(\frac{\lambda_D}{\ell}\right)^2 \tag{A5}$$

using $E_\parallel$. Because the time variation of $\vec{V}$ in the frequency scale $\delta\omega_0$ is included, the two estimates in the above should be enhanced by the factor δ^{-1}.

Thus, the assumption of quasi neutrality is justified if the conditions $\left(\upsilon_A / c\right)^2 << \delta$, $\left(\lambda_D / \ell\right)^2 << \delta$ are satisfied, in addition to the motion being non-relativistic.

APPENDIX B: SOLUTION OF LINEARIZED KINETIC EQUATIONS BY EXPANSION IN SONINE POLYNOMIALS

The methods of solution of a number of linearized kinetic equations by expansion in Sonine polynomials are described in this Appendix.

B.1. Parallel ion velocities and heat flux

Consider first Eq. (84) and Eq. (85), wherein f'_{i1} and f'_{j1} are sums of parts proportional to D_{ij} and $n_{ij}\nabla_\parallel T$. The part proportional to D_{ij} can be expanded in the form

$$f'_{i1} = -\frac{\bar{\tau}_{ij}}{p_{ij}} \upsilon'_\parallel \sum_{n=0}^{N} a_n L_n^{(3/2)}\left(m_i \upsilon'^2/2T\right) f_{i0} D_{ij} \tag{B1}$$

$$f'_{j1} = -\frac{\bar{\tau}_{ij}}{p_{ij}} \upsilon'_\parallel \sum_{n=0}^{N} b_n L_n^{(3/2)}\left(m_j \upsilon'^2/2T\right) f_{j0} D_{ij} \tag{B2}$$

where the coefficients a_n and b_n are dimensionless. Substituting into Eq. (84) and Eq. (85), and integrating over velocity after multiplication by $\upsilon'_\parallel L_m^{(3/2)}\left(m_i \upsilon'^2/2T\right)$ or $\upsilon'_\parallel L_m^{(3/2)}\left(m_j \upsilon'^2/2T\right)$ converts them into the simultaneous linear equations that hold for $m = 0$ to N :

$$\left(\frac{m_j}{m_i}\right)^{1/4} \sum_{n=0}^{N} \left(\frac{Z_i^2 n_i}{Z_j^2 n_{ij}} H_{mn} + \frac{n_j}{n_{ij}} M_{mn}^{ij}\right) a_n + \left(\frac{m_i}{m_j}\right)^{1/4} \frac{n_j}{n_{ij}} \sum_{n=0}^{N} N_{mn}^{ij} b_n = -\frac{n_{ij}}{n_i} \delta_{m0} \tag{B3}$$

$$\left(\frac{m_i}{m_j}\right)^{1/4} \sum_{n=0}^{N} \left(\frac{Z_j^2 n_j}{Z_i^2 n_{ij}} H_{mn} + \frac{n_i}{n_{ij}} M_{mn}^{ji}\right) b_n + \left(\frac{m_j}{m_i}\right)^{1/4} \frac{n_i}{n_{ij}} \sum_{n=0}^{N} N_{mn}^{ji} a_n = \frac{n_{ij}}{n_j} \delta_{m0} \quad . \tag{B4}$$

The matrix coefficients for the Fokker-Planck operator in these equations are defined by

$$\frac{n_a}{\tau_{ab}} M_{mn}^{ab} = \int d^3\upsilon \frac{\upsilon_\parallel}{\bar{\upsilon}_a} L_m^{(3/2)}\left(\frac{\upsilon^2}{\bar{\upsilon}_a^2}\right) C_{ab}\left[\frac{2\upsilon_\parallel}{\bar{\upsilon}_a} L_n^{(3/2)}\left(\frac{\upsilon^2}{\bar{\upsilon}_a^2}\right) f_{a0}, f_{b0}\right] \tag{B5}$$

$$\frac{n_a}{\tau_{ab}} N_{mn}^{ab} = \int d^3\upsilon \frac{\upsilon_\parallel}{\bar{\upsilon}_a} L_m^{(3/2)}\left(\frac{\upsilon^2}{\bar{\upsilon}_a^2}\right) C_{ab}\left[f_{a0}, \frac{2\upsilon_\parallel}{\bar{\upsilon}_b} L_n^{(3/2)}\left(\frac{\upsilon^2}{\bar{\upsilon}_b^2}\right) f_{b0}\right] \tag{B6}$$

where $\bar{\upsilon}_a = \sqrt{2T/m_a}, \bar{\upsilon}_b = \sqrt{2T/m_b}$. They satisfy the relations

$$M_{mn}^{ab} = M_{nm}^{ab}, \quad N_{mn}^{ab} = \sqrt{m_a/m_b} N_{nm}^{ba}, \quad M_{0n}^{ab} + N_{0n}^{ba} = 0 \quad . \tag{B7}$$

Also, $H_{mn} = M_{mn}^{aa} + N_{mn}^{aa}$. The two equations for $m = 0$ are linearly dependent on each other, so that only one needs to be kept. The system is augmented by the equation

$$\frac{n_i}{n_{ij}} a_0 + \frac{n_j}{n_{ij}} b_0 = 0 \quad , \tag{B8}$$

which follows from the requirement $\rho_i u_{\parallel i} + \rho_j u_{\parallel j} = 0$, as contribution of electrons to the mass velocity is of order $\sqrt{m_e/m_i}$ smaller. The transport coefficient l_{11} and l_{21} for the differential flow and total heat flux defined in Eq. (87) and Eq. (88) are calculated from

$$l_{11} = \sqrt{\frac{m_j}{m_i}} a_0 - \sqrt{\frac{m_i}{m_j}} b_0 \tag{B9}$$

$$l_{21} = -\frac{5}{2}\left(\sqrt{\frac{m_j}{m_i}} \frac{n_i}{n_{ij}} a_1 + \sqrt{\frac{m_i}{m_j}} \frac{n_j}{n_{ij}} b_1 \right) \quad . \tag{B10}$$

The parts of the distribution functions arising from $n_{ij} \nabla_\parallel T$ are expanded similarly to Eq. (B1) and Eq. (B2), with $n_{ij} \nabla_\parallel T$ replacing D_{ij}. The coefficients l_{12} and l_{22} are also calculated from Eq. (B9) and Eq. (B10), except the coefficients a_n and b_n now satisfy Eq. (B3) and Eq. (B4) with the right hand sides replaced by $(5/2)\delta_{m1}$ in both.

Defining $r = m_a/m_b$, the matrix coefficients of the Fokker-Planck operator for $m,n = 0,1,2$ are listed below:[8,13]

$$M_{00}^{ab} = -(1+r)^{-1/2} \qquad M_{01}^{ab} = -(3/2)(1+r)^{-3/2} \qquad M_{02}^{ab} = -(15/8)(1+r)^{-5/2}$$

$$M_{11}^{ab} = -\left(13/4 + 4r + 15r^2/2\right)(1+r)^{-5/2} \qquad M_{12}^{ab} = -\left(69/16 + 6r + 63r^2/4\right)(1+r)^{-7/2}$$

$$M_{22}^{ab} = -\left(433/64 + 17r + 459r^2/8 + 28r^3 + 175r^4/8\right)(1+r)^{-9/2}$$

$$N_{00}^{ab} = r^{1/2}(1+r)^{-1/2} \qquad N_{01}^{ab} = \left(3r^{3/2}/2\right)(1+r)^{-3/2} \qquad N_{02}^{ab} = \left(15r^{5/2}/8\right)(1+r)^{-5/2}$$

$$N_{10}^{ab} = \left(3r^{1/2}/2\right)(1+r)^{-3/2} \qquad N_{11}^{ab} = \left(27r^{3/2}/4\right)(1+r)^{-5/2} \qquad N_{12}^{ab} = \left(225r^{5/2}/16\right)(1+r)^{-7/2}$$

$$N_{20}^{ab} = \left(15r^{1/2}/8\right)(1+r)^{-5/2} \qquad N_{21}^{ab} = \left(225r^{3/2}/16\right)(1+r)^{-7/2} \qquad N_{22}^{ab} = \left(2625r^{5/2}/64\right)(1+r)^{-9/2} \quad .$$

From these the elements of the symmetric matrix H can be calculated to give

$$H_{00} = H_{01} = H_{02} = 0$$

$$H_{11} = -\sqrt{2} \qquad H_{12} = -3\sqrt{2}/4 \qquad H_{22} = -45\sqrt{2}/16 \quad .$$

Figure 1(a-c) shows examples of the coefficients l_{ij} as functions of n_i/n_{ij} for the case $m_i/m_j = 3/2, Z_i/Z_j = 1$, such as when the two ion components are tritons and deuterons. They are obtained using three terms in the expansions in Sonine polynomials. The coefficient l_{11} diverges as n_i/n_{ij} tends to zero or one. This does not present a problem as D_{ij} approaches zero in these limits.

The limit $n_i/n_{ij} \to 0$ corresponds to a trace amount of the ion species a in a simple plasma with species b. In this case, $u_{\parallel j} = 0$ and Eq. (87) expresses $u_{\parallel i}$ as the sum of a diffusive and a convective flow. Also, the coefficient l_{22} corresponds to the ion thermal conductivity for a simple plasma given in Eq. (56).

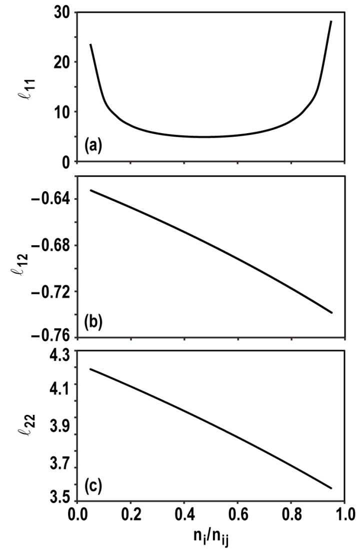

FIG. 1. Transport coefficients for parallel differential ion velocity and total heat flux ($m_i/m_j = 3/2, Z_i = Z_j = 1$).

B.2. Parallel electron velocity and heat flux

The electron parallel transport equation [Eq. (50)] can be similarly solved with the expansion

$$f'_{e1} = -\tau_{ei} \upsilon'_{\parallel} \sum_{n=0}^{N} c_n L_n^{(3/2)} \left(m_e \upsilon'^2 / 2T \right) f_{e0} \left(\nabla_{\parallel} p_e + n_e e E_{\parallel}^{(0)} \right) / p_e \tag{B11}$$

for the part proportional to $\nabla_{\parallel} p_e + n_e e E_{\parallel}^{(0)}$. The coefficients c_n satisfy the equation

$$\sum_{n=0}^{N} \left(Z_{eff}^{-1} H_{mn} + L_{mn} \right) c_n = -\delta_{m0} \tag{B12}$$

where $L_{mn} = M_{mn}^{ab}$ with $r = 0$ corresponds to the pitch angle scattering operator, and $Z_{eff} = \left(Z_i^2 n_i + Z_j^2 n_j \right) / n_e$. The transport coefficients defined in Eqs. (52) and (53) are determined from

$$\lambda_{11} = c_0 \qquad \lambda_{21} = -5c_1/2 \quad . \tag{B13}$$

The coefficients λ_{12} and λ_{22} for the part proportional to $\nabla_{\parallel} T$ are similarly calculated, except Eq. (B12) for c_n now has $5\delta_{m1}/2$ on the right hand side. The values of $\lambda_{11}, \lambda_{12}$ and λ_{22} for some values of Z_{eff} can be found from Table B1. They agree well with those in Ref. 8.

B.3. Parallel ion viscosity

For the computation of the ion contribution to parallel viscosity in a two-ion species plasma, the ion version of Eq. (36) can be written in the form

$$C_{ii}\left(f'_{i1},f_{i0}\right)+C_{ii}\left(f_{i0},f'_{i1}\right)+C_{ij}\left(f'_{i1},f_{j0}\right)+C_{ij}\left(f_{i0},f'_{j1}\right)+C_{ie}\left(f'_{i1},f_{e0}\right)=\frac{m_i\upsilon'^2}{2T}P_2\left(\xi\right)f_{i0}\hat{b}\hat{b}:\vec{\vec{W}}_0 \quad \text{(B14)}$$

$$C_{jj}\left(f'_{j1},f_{j0}\right)+C_{jj}\left(f_{j0},f'_{j1}\right)+C_{ji}\left(f'_{j1},f_{i0}\right)+C_{ji}\left(f_{j0},f'_{i1}\right)+C_{je}\left(f'_{j1},f_{e0}\right)=\frac{m_j\upsilon'^2}{2T}P_2\left(\xi\right)f_{j0}\hat{b}\hat{b}:\vec{\vec{W}}_0 \quad . \text{(B15)}$$

Here the ion-electron collision terms C_{ie}, C_{je} are given by Eq. (71). We have neglected $C_{ie}\left(f_{i0},f'_{e1}\right)$ and $C_{je}\left(f_{j0},f'_{e1}\right)$ because they are of higher order in the electron to ion mass ratio compared with the terms retained in view of the estimate $f'_{e1}/f'_{i1}\sim f'_{e1}/f'_{j1}\sim\sqrt{m_e/m_i}$. The system of equations is therefore decoupled from the electron equation. It is solved by making the expansions

$$f'_{i1}=x_i^2P_2\left(\xi\right)\sum_{n=0}^{N}a_nL_n^{(5/2)}\left(x_i^2\right)f_{i0}\bar{\tau}_{ij}\hat{b}\hat{b}:\vec{\vec{W}}_0 \quad \text{(B16)}$$

$$f'_{j1}=x_j^2P_2\left(\xi\right)\sum_{n=0}^{N}b_nL_n^{(5/2)}\left(x_j^2\right)f_{j0}\bar{\tau}_{ij}\hat{b}\hat{b}:\vec{\vec{W}}_0 \quad , \quad \text{(B17)}$$

where $x_i^2=m_i\upsilon'^2/2T$, $x_j^2=m_j\upsilon'^2/2T$. The dimensionless coefficients a_n,b_n satisfy the linear equations

$$\left(\frac{m_j}{m_i}\right)^{1/4}\sum_{n=0}^{N}\left(\frac{Z_i^2n_i}{Z_j^2n_{ij}}H'_{mn}+\frac{n_j}{n_{ij}}M'^{ij}_{mn}+\frac{n_e}{Z_j^2n_{ij}}M'^{ie}_{mn}\right)a_n+\left(\frac{m_i}{m_j}\right)^{1/4}\frac{n_j}{n_{ij}}\sum_{n=0}^{N}N'^{ij}_{mn}b_n=\frac{3}{4}\delta_{m0} \quad \text{(B18)}$$

$$\left(\frac{m_i}{m_j}\right)^{1/4}\sum_{n=0}^{N}\left(\frac{Z_j^2n_j}{Z_i^2n_{ij}}H'_{mn}+\frac{n_i}{n_{ij}}M'^{ji}_{mn}+\frac{n_e}{Z_i^2n_{ij}}M'^{je}_{mn}\right)b_n+\left(\frac{m_j}{m_i}\right)^{1/4}\frac{n_i}{n_{ij}}\sum_{n=0}^{N}N'^{ji}_{mn}a_n=\frac{3}{4}\delta_{m0} \quad , \quad \text{(B19)}$$

where we have introduced the matrix elements of the Fokker-Planck operator as follows:

$$\frac{n_a}{\tau_{ab}}M'^{ab}_{mn}=\int d^3\upsilon x_a^2L_m^{(5/2)}\left(x_a^2\right)P_2\left(\xi\right)C_{ab}\left[x_a^2L_n^{(5/2)}\left(x_a^2\right)P_2\left(\xi\right)f_{a0},f_{b0}\right] \quad \text{(B20)}$$

$$\frac{n_a}{\tau_{ab}}N'^{ab}_{mn}=\int d^3\upsilon x_a^2L_m^{(5/2)}\left(x_a^2\right)P_2\left(\xi\right)C_{ab}\left[f_{a0},x_b^2L_n^{(5/2)}\left(x_b^2\right)P_2\left(\xi\right)f_{b0}\right] \quad \text{(B21)}$$

$$H'_{mn} = M'^{aa}_{mn} + N'^{aa}_{mn} \quad . \tag{B22}$$

The contribution of ions to the parallel viscous stress is given by

$$\vec{\vec{\pi}}_{\|i} + \vec{\vec{\pi}}_{\|j} = \left(p_i a_0 + p_j b_0\right) \overline{\tau}_{ij} \vec{\vec{W}}_0 \quad . \tag{B23}$$

The values of the matrix elements in Eq. (B20) and Eq. (B21) for $m,n = 0,1$ are listed below, where $r = m_a/m_b$:

$$M'^{ab}_{00} = -\left(1+r\right)^{-3/2}\left(9/10 + 3r/2\right) \qquad M'^{ab}_{01} = M'^{ab}_{10} = -\left(1+r\right)^{-5/2}\left(27/20 + 63r/20\right)$$

$$M'^{ab}_{11} = -\left(1+r\right)^{-7/2}\left(153/40 + 555r/40 + 231r^2/20 + 21r^3/2\right)$$

$$N'^{ab}_{00} = \left(1+r\right)^{-3/2}\left(3r/5\right) \quad N'^{ab}_{01} = \left(1+r\right)^{-5/2}\left(9r^2/5\right) \quad N'^{ab}_{10} = \left(1+r\right)^{-5/2}\left(9r/5\right)$$

$$N'^{ab}_{11} = \left(1+r\right)^{-7/2}\left(9r^2\right)$$

$$H'_{00} = -9\sqrt{2}/20 \quad H'_{01} = H'_{10} = -27\sqrt{2}/80 \quad H'_{11} = -123\sqrt{2}/64 \quad .$$

The matrix elements of the ion-electron collision operator are taken in the leading order in $\sqrt{m_e/m_i}$:

$$M'^{ie}_{00} = -\left(3/2\right)\sqrt{m_e/m_i} \qquad M'^{ie}_{01} = M'^{ie}_{10} = 0 \quad M'^{ie}_{11} = -\left(21/2\right)\sqrt{m_e/m_i} \quad .$$

Treating the ion-electron collision terms as small perturbations, the coefficients a_0 and b_0 are linearly dependent on $\sqrt{m_e/m_i}$ and $\sqrt{m_e/m_j}$, as expressed in Eq. (99). Figure 2(a-f) shows examples of the coefficients $\alpha_i, \beta_i, \gamma_i, \alpha_j, \beta_j, \gamma_j$ as functions of n_i/n_{ij} for the case $m_i/m_j = 3/2, Z_i = 1, Z_j = 1$.

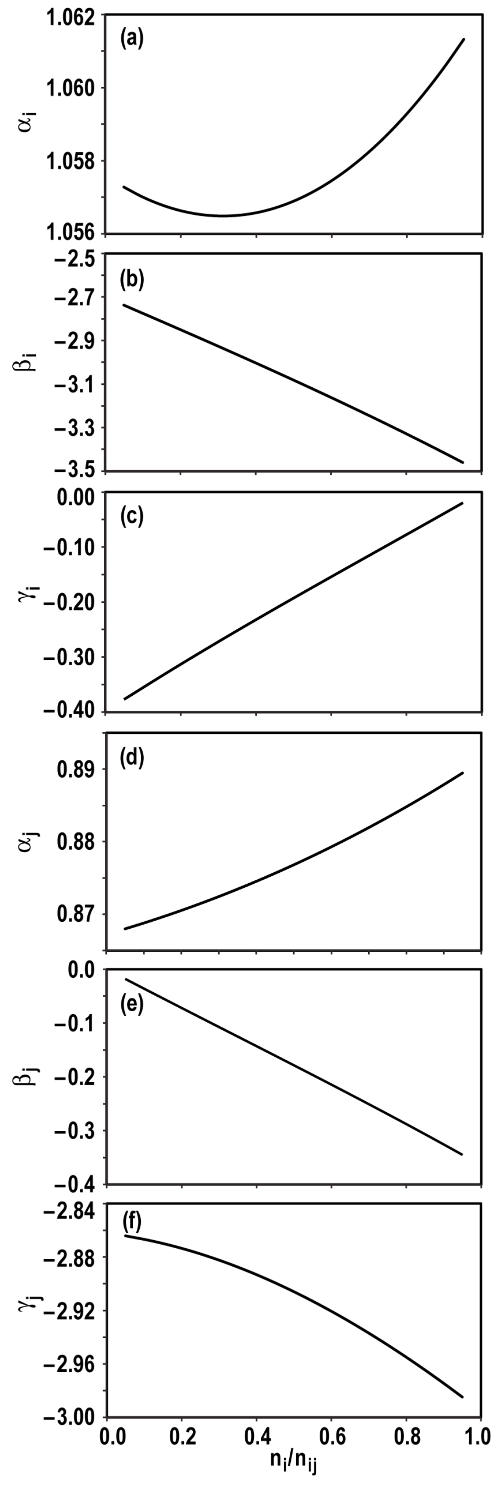

FIG. 2. Parallel viscosity coefficients ($m_i/m_j = 3/2, Z_i = Z_j = 1$).

The case of plasma with a single ion species is obtained by setting n_j to zero.

B.4. Parallel electron viscosity

The electron contribution to the parallel viscous stress is obtained from the solution of the equation

$$C_{ee}\left(f'_{e1}, f_{e0}\right) + C_{ee}\left(f_{e0}, f'_{e1}\right) + \nu_{eij} L f'_{e1} = \frac{m_e \upsilon'^2}{2T} P_2\left(\xi\right) f_{e0} \hat{b}\hat{b} : \vec{\vec{W}} \quad , \tag{B24}$$

where $\nu_{eij} = 4\pi Z_{eff} n_e e^4 \ell n\Lambda / m_e \upsilon^3$. The expansion

$$f'_{e1} = x_e^2 P_2\left(\xi\right) \sum_{n=0}^{N} c_n L_n^{(5/2)}\left(x_e^2\right) f_{e0} \tau_{ee} \hat{b}\hat{b} : \vec{\vec{W}}_0 \tag{B25}$$

turns it into the equation

$$\sum_{n=0}^{N} \left(H'_{mn} + Z_{eff} L'_{mn}\right) c_n = \frac{3}{4} \delta_{m0} \quad , \tag{B26}$$

where $L'_{mn} = M'^{aa}_{mn}$ with $r = 0$ are the matrix elements of the pitch-angle-scattering operator. The parallel viscous stress takes the form in Eq. (68), with

$$\alpha_e = -c_0 \quad . \tag{B27}$$

The values of α_e for a number of Z_{eff} can be found in Table B1.

Table B1. Electron parallel transport coefficients

Z_{eff}	λ_{11}	λ_{12}	λ_{22}	α_e
1	1.95	1.39	4.15	0.73
2	2.32	2.10	6.79	0.51
4	2.67	2.91	10.10	0.32
16	3.13	4.27	16.14	0.10
∞	3.39	5.16	20.31	0.00

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