A Perturbation Solution to the Drift Kinetic Equation Yields Pinch Type Fluxes From the Circulating Electrons.

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General Atomics

Introduction

- Particle Flux Pinch Terms proportional to $\nabla q/q$ and $\nabla T/T$ can be obtained from the Drift Kinetic Equation

- These Pinch Terms can come from Trapped Electrons
  (See Garbet and Angioni – this conference)

- If the Turbulence has a Ballooning Character then Passing Electrons can also contribute

Outline

- **Present an Expression for the Particle Flux**
- **Use an iterative technique to solve for $\tilde{f}$ from DKE**
- **Key Factor is to take the Gradients Correctly in DKE**
- **Velocity integrate $\tilde{f}$ to get $\tilde{n}$ with Pinch Like Terms**
- **Discuss Relative importance of $\nabla q/q$ and $\nabla T/T$ Terms**

Expression for Particle Flux

For Low Frequency Turbulence, the Particle Flux can be Written as,

$$\Gamma = \int d^3v \left\langle \left\langle f\, \vec{\mathbf{v}}_d \cdot \hat{\rho} \right\rangle_{ens} \right\rangle_{fs} ,$$

with $\hat{\rho} \equiv \nabla\rho / \left|\nabla\rho\right|$ and ρ is the usual minor radius. $\vec{\mathbf{v}}_d$ is the particle drift velocity.

The Particle Flux has Two Terms

$$\Gamma = \Gamma_a + \Gamma_b = \int d^3v \left\langle f_1 \vec{\mathbf{v}}_{d0} \cdot \hat{\rho} \right\rangle_{fs} + \int d^3v \left\langle \left\langle \tilde{f}\, \tilde{\mathbf{v}}_{\mathrm{E}} \cdot \hat{\rho} \right\rangle_{ens} \right\rangle_{fs} .$$

Γ_a Contains Pinch Terms, but may be Forced to be Zero

It has been shown that $\Gamma_a \equiv \int d^3 \mathrm{v} \left\langle f_1 \vec{\mathbf{v}}_{d0} \cdot \nabla\psi \right\rangle_{fs}$ results in pinch terms ~ ∇q/q.

Baker, Phys. Plasmas 9, 2675 (2002)

But this term is not intrinsically ambipolar and may thus be forced to be zero by electrostatic forces.

Note: Γ_a has the form $\Gamma_a \propto -D\left(\frac{\partial n}{\partial \psi} + \alpha h \frac{1}{q} \frac{\partial q}{\partial \psi}\right), \quad \alpha \sim 1$, thus even if it is forced to be zero, it still may impose constraints upon the density profile.

Here We Look for Pinch Terms in $\langle \tilde{n}\vec{\mathbf{v}}_E \rangle_{ens}$

Consider $\square_b \equiv \square d^3\mathrm{v} \left\langle \left\langle \tilde{f}\,\tilde{\mathbf{v}}_E \right\rangle_{ens} \cdot \square \right\rangle_{fs} = \left\langle \left\langle \tilde{n}\,\tilde{\mathbf{v}}_E \right\rangle_{ens} \cdot \square \right\rangle_{fs}$

This term is intrinsically ambipolar

We will need a nonzero ensemble average for $\left\langle \tilde{n}\,\tilde{\mathbf{v}}_E \right\rangle_{ens} \cdot \square$

Since $\tilde{\mathbf{v}}_{\mathrm{E}} \cdot \square \equiv \dfrac{c\tilde{\mathbf{E}} \square \vec{\mathbf{B}}}{B^2} \cdot \square \square \square i \dfrac{ck_{\square}\tilde{\square}_k}{B}$,

we will then need $\tilde{n} \quad i\,k_{\square}\tilde{\square}_k$.

Obtain Solution from Drift Kinetic Equation (for electrons)

Start with the drift kinetic equation

$$\frac{\partial \bar{f}}{\partial t} + \left(\vec{\mathbf{v}}_D + \vec{\mathbf{v}}_{\parallel}\right) \cdot \nabla \bar{f} + Ze\left(\vec{\mathbf{v}}_D + \vec{\mathbf{v}}_{\parallel}\right) \cdot \vec{\mathbf{E}} \frac{\partial \bar{f}}{\partial K} = C\left(\bar{f}\right) \approx 0$$

The fluctuating part can be written as (assuming that $\vec{\mathbf{v}}_{ExB} = \tilde{\mathbf{v}}_E$),

$$\frac{\partial \tilde{f}}{\partial t} + \left(\vec{\mathbf{v}}_d + \vec{\mathbf{v}}_{\parallel}\right) \cdot \nabla \tilde{f} + \tilde{\mathbf{v}}_E \cdot \nabla f_M - Ze\left(\vec{\mathbf{v}}_d + \vec{\mathbf{v}}_{\parallel}\right) \cdot \nabla \tilde{\phi} \frac{\partial f_M}{\partial K} = 0$$

$\vec{\mathbf{v}}_d$ **contains the curvature and gradient drifts.** *K* **is the kinetic energy.**

In order to calculate the fluxes, one must perform the gradients correctly.

$$\nabla \tilde{f} = \nabla_{\parallel} \tilde{f} + \nabla_{\perp} \tilde{f} \approx \left[i \vec{\mathbf{k}} \tilde{f} + \hat{\mathbf{b}} \partial \tilde{f} / \partial \ell + \nabla \psi \partial \tilde{f} / \partial \psi \right].$$

Fluctuating part of Drift Kinetic Equation

$$\frac{\partial \tilde{f}}{\partial t} + \left(\vec{\mathbf{v}}_d + \vec{\mathbf{v}}_{\parallel}\right)\cdot\nabla\tilde{f} + \tilde{\mathbf{v}}_E \cdot \nabla f_M - Ze\left(\vec{\mathbf{v}}_d + \vec{\mathbf{v}}_{\parallel}\right)\cdot\nabla\tilde{\phi}\frac{\partial f_M}{\partial K} = 0 .$$

Fourier transform in time and space, then

$$-\mathrm{i}\left(\omega - \omega_d\right)\tilde{f}_k + \omega_{\square}\tilde{f}_k + \mathrm{v}_{\parallel}\frac{\partial \tilde{f}_k}{\partial \ell} = \mathrm{i}\left(\omega_* - \omega_d\right)\frac{\mathrm{Ze}\tilde{\phi}_k}{\mathrm{T}} f_M - \mathrm{v}_{\parallel}\frac{\partial}{\partial \ell}\left(\frac{\mathrm{Ze}\tilde{\phi}_k}{\mathrm{T}}\right) f_M ,$$

where we have neglected $\vec{\mathbf{k}}_{\parallel}$ **and defined** $\omega_{\square} \equiv \left(\vec{\mathbf{v}}_d \cdot \tilde{\nabla}\frac{\partial}{\partial \square}\right)$.

We neglect $\vec{\mathbf{k}}_{\parallel}$, because including it does not change the basic result and neglecting it simplifies the solution.

$\omega_{\square}$ **&** $\partial/\partial\ell$ **operate on the magnitude, not the phase**

We can write

$$v_{\parallel}\frac{\partial \tilde{f}_k}{\partial \ell} = -v_{\parallel}\frac{\partial}{\partial \ell}\left(\frac{\mathrm{Ze}\tilde{\phi}_k}{\mathrm{T}}\right) f_M + \mathrm{i}\left(\omega_* - \omega_d\right)\frac{\mathrm{Ze}\tilde{\phi}_k}{\mathrm{T}} f_M + \mathrm{i}\left(\omega - \omega_d\right)\tilde{f}_k - \nu \tilde{f}_k$$

Solving for the perturbed distribution function for electrons (with $\mathbf{v}_{\parallel}$ large)

$$\tilde{f}_k = -\frac{Ze\tilde{\phi}_k}{T} f_M - i\int d\ell \frac{\left(\omega - \omega_*\right)}{v_{\parallel}}\frac{\mathrm{Ze}\tilde{\phi}_k}{\mathrm{T}} f_M - \int d\ell \frac{\vec{\mathbf{v}}_d \cdot \nabla\psi}{v_{\parallel}}\frac{\partial \tilde{f}_k}{\partial \psi} + \ldots.$$

Only the last term contributes to transport since the $\mathbf{v}_{\parallel}$ in the denominator of the 2nd term velocity integrates to zero. Using a recursion technique,

$$\tilde{f}_k(with\ \ the\ \ right\ \ phase) \equiv \tilde{f}_{90} \approx i\int d\ell \frac{\vec{\mathbf{v}}_d \cdot \nabla\psi}{v_{\parallel}}\frac{\partial}{\partial \psi}\left[\int d\ell \frac{\left(\omega - \omega_*\right)}{v_{\parallel}}\frac{\mathrm{Ze}\tilde{\phi}_k}{\mathrm{T}} f_M\right]$$

Note: By definition, $\partial/\partial\psi$ acts upon the magnitude and not upon the phase.

Evaluate $\tilde{f}_{90}$ to obtain $\tilde{n}$

The fluctuating part of the distribution function which has the right phase and velocity dependence to result in net transport is

$$\tilde{f}_{90} \approx -i\int d\ell \frac{\vec{\mathbf{v}}_d \cdot \nabla\psi}{\mathrm{v}_{\|}} \frac{\partial}{\partial\psi}\left[\int d\ell \frac{\left(\omega - \omega_*\right)}{\mathrm{v}_{\|}} \frac{\mathrm{Ze}\tilde{\phi}_k}{\mathrm{T}} f_M\right]$$

Need approximate expression for $\oint d\ell$. We know,

$$\oint \frac{d\ell}{B} = \frac{1}{2\pi}\frac{\partial Vol}{\partial \psi} = \frac{2\pi R_0}{B_0} qH . \qquad \Rightarrow \; d\ell \approx R_0 qH \, d\theta . \quad \left(H \equiv \frac{B_{T0}}{2\pi R_0}\frac{\partial Vol}{\partial \psi} \right)$$

Since the turbulence has a ballooning character, assume

$$\tilde\phi_k \approx \tilde\phi_0(k,\psi)\left[1+\varepsilon_b \cos\theta\right]$$

Note: $\vec{\mathbf{v}}_d \cdot \nabla\tilde\phi = \left(\frac{2T}{m\Omega}\right)\hat{\mathbf{b}}\times\boldsymbol{\kappa}\cdot\nabla\tilde\phi \equiv i\left(\frac{2T}{m\Omega}\right)\frac{F_d(\theta)}{R}, \quad with \;\; F_d(\theta) \sim sin(\theta)$

Assume that ω and ω_* are approximately independent of θ, then

$$\tilde f_{90} \approx iR_0 qH \int d\theta \frac{\vec{\mathbf{v}}_d \cdot \nabla\tilde\phi_k}{\mathrm{v}_\parallel}\frac{\partial}{\partial\theta}\left[\frac{R_0 qH}{\tilde\phi_k}\frac{\left(\omega-\omega_*\right)}{\mathrm{v}_\parallel} f_M \int d\theta \frac{\mathrm{Ze}\tilde\phi_k}{\mathrm{T}}\right]$$

$$\tilde f_{90} \approx -i\frac{\mathrm{ZeR_0^2}qH}{\left(1+\varepsilon_b \cos\theta\right)}\frac{2T}{m\Omega}\left[\int^{\theta} d\theta \sin\theta \int_{\theta_{b.c.}}^{\theta} d\theta\left(1+\varepsilon_b\cos\theta\right)\right]\left[\frac{\left(\omega-\omega_*\right)}{\mathrm{T}}\frac{\tilde\phi_k}{\mathrm{v}_\parallel}\frac{\partial}{\partial\theta}\left(qH\frac{f_M}{\mathrm{v}_\parallel}\right)\right]$$

Because of the ballooning character of the turbulence, $(1 + \epsilon_b \cos\theta)$ term, the integrals over theta result in a function of theta which does not flux surface to zero.

We have not yet explicitly included the effects of trapping. Including trapping, effectively changes the limits of the theta integrals from $-\pi$ to π, to $-\theta_{turn}$ to $\theta < \theta_{turn}$.

Thus trapped electrons will produce in a non zero result even with non ballooning turbulence.

The turning angle for the trapped electrons, θ_{turn}, depends upon the pinch angle at the outer midplane and thus depends upon velocity. Including this is straight forward but complicates the result without qualitatively changing the result.

Here we ignore trapping effects.

$$\tilde{f}_{90} \approx -i\tilde{\phi}_k e R_0^2 |F(\theta)| \frac{2T}{m|\omega|} \frac{(\omega-\omega_*)}{T} \frac{qH}{v_\parallel} \frac{\partial}{\partial\theta}\left(qH \frac{f_M}{v_\parallel} \right)$$

For electrons, $\frac{\omega-\omega_*}{T} = \frac{\omega-\omega_{*e}}{T_e} = \frac{\omega_{*e}}{T_e}\left[\frac{\omega}{\omega_{*e}} - 1 - \left(\frac{K}{T_e} - \frac{3}{2} \right) \frac{n_e}{T_e}\frac{\partial T_e}{\partial n_e} \right]$,

and

$$\tilde{f}_{90} \approx i k_\theta \tilde{\phi}_k e R_0^2 \frac{2T_e}{m_e^2 \Omega_e^2} \left[\left(1 - \frac{\omega}{\omega_{*e}} \right) \frac{1}{n_e}\frac{\partial n_e}{\partial\psi} + \left(\frac{K}{T_e} - \frac{3}{2} \right) \frac{1}{T_e}\frac{\partial T_e}{\partial\psi} \right] \frac{qH}{v_\parallel} \frac{\partial}{\partial\theta}\left(qH \frac{f_M}{v_\parallel} \right)$$

Now need to do velocity integral. Simplest thing to do is to replace *K* by T_e and $v_\parallel$ by $(T_e/m_e)^{1/2}$. Then

$$\tilde{n}_{90} \sim i k_\theta \tilde{\phi}_k \left[\left(1 - \frac{\omega}{\omega_{*e}} \right) \frac{1}{n_e}\frac{\partial n_e}{\partial\psi} - \frac{1}{2}\frac{1}{T_e}\frac{\partial T_e}{\partial\psi} \right] \frac{qH}{T_e^{1/2}} \frac{\partial}{\partial\theta}\left(\frac{n_e qH}{T_e^{1/2}} \right)$$

Can rewrite as,

$$\tilde{n}_{90} \sim \square i k_{\square} \tilde{\square}_k \frac{(qH)^2}{\mathrm{T_e} L_n} [1 \square \square\square] \left[\frac{1}{n_e} \frac{\partial n_e}{\partial \square} + \frac{1}{qH} \frac{\partial qH}{\partial \square} \square \frac{1}{2T_e} \frac{\partial T_e}{\partial \square} \right]$$

where, $\square \equiv 1/2\left(1 \square \square/\square_{*e}\right)$ **and** $\square \equiv \left(n_e/T_e\right)\left(\partial T_e/\partial n_e\right) \equiv L_n/L_T$

References for pinch related to (1-$\square\square$)
G.S. Lee and P.H. Diamond, Phys. Fluids 29, (1986) p3291
K.C. Shaing, Phys. Fluids 31, (1988) p 2249
P.W. Terry, Phys. Fluids B 1, (1989) p 193

or (for illustrative purposes) we can rewrite as,

$$\tilde{n}_{90} \sim \square i k_{\square} \tilde{\square}_k \frac{(qH)^2}{\mathrm{T_e} L_n} \left[\frac{1}{n_e} \frac{\partial n_e}{\partial \square} + \frac{1}{qH} \frac{\partial qH}{\partial \square} \square \left(\frac{1}{2} + \square \left(1 \square \frac{L_n}{L_{qH}} \square \frac{L_n}{2L_T} \right) \right) \frac{1}{T_e} \frac{\partial T_e}{\partial \square} \right]$$

This shows the form of the final solution (after doing the velocity integrals)

Perform Velocity Integrals

(Need to eliminate the singularity at $v_{||} = 0.0$)

Go Back to Approximate Solution for $\tilde{f}$

$$\tilde{f}_{90} \approx -i\,k_{\theta}\tilde{\phi}_k \mathrm{eR}_0^2 \frac{2T_e}{m_e^2\Omega_e^2}\left[\left(1-\frac{\omega}{\omega_{*e}}\right)\frac{1}{n_e}\frac{\partial n_e}{\partial \psi}+\left(\frac{K}{T_e}-\frac{3}{2}\right)\frac{1}{T_e}\frac{\partial T_e}{\partial \psi}\right]\frac{qH}{\mathrm{v}_{\|}}\frac{\partial}{\partial \theta}\left(qH\frac{f_M}{\mathrm{v}_{\|}}\right)$$

In order to do the velocity integral it is necessary to eliminate the singularity at $\mathrm{v}_{\|} = 0$. Since the approximate solution is only valid for 'large' $\mathrm{v}_{\|}$, this is appropriate and can be done in an approximate manner.

Solving for $\tilde{f}$ for $\mathrm{v}_{\|} \to 0.0$ gives a result for $\tilde{f}$ which flux surface averages to zero. Therefore one approximate solution for $\tilde{n}$ would be to set $\tilde{f}$ to 0.0 for $|\mathrm{v}_{\|}| < \mathrm{v}_{\|\mathrm{min}}$. With $\mathrm{v}_{\|\mathrm{min}} \approx \mathrm{q}H\,\mathrm{C_S}$. ($\mathrm{C_S}$ is the sound speed)

There are other approximate solutions for small $\mathrm{v}_{\|}$ which can be used.

The approximate solution for $\tilde{n}$ then becomes,

$$\tilde{n}_{90} \sim \square\, i\, k_{\square} \tilde{\square}_k \frac{(qH)}{\mathrm{T_e} L_n} \left(\frac{1}{n_e}\frac{\partial n_e}{\partial \square} + \square_0 \frac{1}{qH}\frac{\partial qH}{\partial \square} \right) \square_1 + \left(\square_2 \square \square_3 \frac{L_n}{L_{qH}} \square \square_4 \frac{L_n}{L_T} \right) \left(\frac{1}{T_e}\frac{\partial T_e}{\partial \square} \right)$$

where the $\square$'s are order 1 **This term can be + , $\square$ or 0**

The values of $\square$ depend upon the approximate solution for $\tilde{f}$

The value of this term depends upon $\square(\square, \ldots)$ and the $\square$'s.

$$\square_b = \left\langle \left\langle \tilde{n}\, \tilde{\mathbf{v}}_{\mathrm{E}} \right\rangle_{ens} \cdot \tilde{\square} \right\rangle_{fs}, \quad \textbf{with} \quad \tilde{\mathbf{v}}_E \cdot \tilde{\square} \square \square i \frac{c k_{\square} \tilde{\square}_k}{B}$$

$$\square_b \sim \square \sum_k \left| k_{\square} \tilde{\square}_k \right|^2 \left(\frac{1}{n_e}\frac{\partial n_e}{\partial \square} + \square_0 \frac{1}{qH}\frac{\partial qH}{\partial \square} \right) \square_1 + \left(\square_2 \square \square_3 \frac{L_n}{L_{qH}} \square \square_4 \frac{L_n}{L_T} \right) \left(\frac{1}{T_e}\frac{\partial T_e}{\partial \square} \right)$$

As an aside

□ contains all of the information about the dispersion relation (if one can be obtained in the fully saturated nonlinear state.) Therefore □ depends upon a range of plasma parameters such as, temperature, density, and gradient scale lengths including □T_i/T_i , as well as L_n, L_{qH} and L_T.

Conclusions

- Both $\int d^3 v \langle f_1 \vec{\mathbf{v}}_{d0} \cdot \vec{\nabla} \rangle_{fs}$ *and* $\langle \langle \tilde{n}\, \tilde{\mathbf{v}}_{\mathrm{E}} \rangle_{ens} \cdot \vec{\nabla} \rangle_{fs}$ result in pinch terms

- When modeling Turbulent Transport it is necessary to take derivatives correctly (i.e. $\nabla \neq i\,\vec{\mathbf{k}}$)

- The Flux Proportional to ∇**q/q** is always Inward, but the Flux Proportional to ∇**T/T** can be Inward, Outward or Zero.

- We need to do experiments and modeling to determine if pinch is determined by **Te** or **q**. **(See Garbet and Angioni – this conference)**

  - A necessary step for the understanding of turbulent transport