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A BRIEF REVIEW OF MAGNETIC WELLS

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Abstract

The notion of ‘‘magnetic well’’ is useful as a figure of merit for tokamak operation. The object of this review is to collect and compare a variety of different forms that have been used as a well parameter. A particularly satisfying general formulation, based on a resistive stability criterion, is presented in Eq. (25). The application of these results to tokamaks is discussed in terms of the well-known analytic, near-axis approximation that displays the importance of the safety factor and of shaping.

The idea of ‘‘magnetic well’’ has had a rather convoluted history, and each phase has left some debris. This note is an effort to organize these developments, bring together a number of approximations, and find a satisfying formulation in Eq. (25). There are several possible reasons for the existence of several slightly different formulae, the most important of which is that magnetic wells have rarely been employed to explain experimental results in tokamaks. With improved diagnostics, and somewhat more venturesome choices of non circular equilibrium shapes, it may be a good time to review the various theories, more or less in historical order.

The basic idea behind this effort was to find a principle analogous to the stable stratification of a fluid. For example, it is clear that if the atmosphere has an inversion, there is little stirring of the air, but if there is no inversion it is well stirred. This can be reduced to the principle that if the entropy density increases with altitude, the atmosphere is stable, otherwise it is marginal or unstable to convection cells of all scales. The demonstration of this result proceeds by considering slowly convecting cells that remain close to static force balance at all times, so that sound waves are not excited, while convecting sufficiently rapidly that entropy is fixed in the fluid. The work done against gravity maintaining such a convection determines stability. The condition that sound waves not be generated by the convection can be crudely expressed as the requirement that the motion be incompressible. However, this idea is misleading. For example, vertical convection must be accompanied by compression or expansion that is precisely determined from the scale height of the atmosphere.

The first effort in this direction for magnetic confinement systems was that of Rosenbluth and Longmire.[1] They considered the forces driving convection cells in a simple magnetic confinement system, following the procedure that is used in treating the atmosphere. This convection carries the magnetic field lines with it, while producing minimal changes in the field configuration. The conclusion that they came to was that magnetic field strength was the important quantity. If the field was weakest where the plasma pressure was highest, the configuration was stably stratified. In this formulation the gradient of the magnetic field strength plays the role that gravity plays in the atmosphere, and the plasma pressure is analogous to the atmospheric entropy. This analogy leads the simple identification of the magnetic field as an equivalent gravitational well, with the well parameter given by Freidberg[2] in Eq. (4.39) as

$$\hat{W} = 2\frac{V}{\left\langle B^2 \right\rangle}\ \frac{d}{dV}\left\langle \frac{B^2}{2} \right\rangle \quad . \tag{1}$$

In this note the configuration is assumed to be basically toroidal and axisymmetric, with the pressure constant on toroidal magnetic surfaces. The usual weighted magnetic surface average is denoted by brackets,

$$\langle * \rangle = \oint * \frac{d\theta}{\boldsymbol{B}\cdot\nabla\theta} \Big/ \oint \frac{d\theta}{\boldsymbol{B}\cdot\nabla\theta} \quad , \tag{2}$$

where θ is any angular coordinate that covers the magnetic surface, and the integrals are taken over a complete loop. The configuration is taken to be stably stratified if

$$p'(V)\,\hat{W} < 0 \quad , \tag{3}$$

where, in this note, prime denotes derivative with respect to the flux surface label in current use, which in the equations above is the contained volume *V*. Here we maintain flexibility, and express the various quantities in forms that are invariant, or essentially invariant, under change of flux surface label. Note that changing the flux surface label in Eq. (3), without altering the leading *V* in Eq. (1), results in p' and $\hat{W}$ each being multiplied by identical quantities, so Eq. (3) is multiplied by a positive quantity and remains valid. The content of Eq. (3) is that the configuration is stably stratified if the pressure *p* and surface averaged magnetic field *B* increase in opposite directions. In particular, we are primarily interested in configurations in which the pressure decreases away from the magnetic axis, so that $p'(V)$ is negative. Thus systems with positive well, $\hat{W} > 0$, are favorable for containment. The important factor in Eq. (1) is the derivative, the rest is a rather arbitrary normalization that will be further discussed below.

When the energy principle became available,[3] it clarified the requirements on the details of the convection cells so that none of the three MHD waves are excited. In particular, the convection should preserve the value of $\mu_0 p + B^2/2$ so that magnetosonic waves are not excited, the convection rolls should be aligned with field lines so that Alfvén waves are not generated, and the plasma pressure should be held constant along each field line so that slow sound waves are absent. As with atmospheric convection, these requirements are crudely but somewhat misleadingly equivalent to an incompressibility constraint.

When these conditions were properly treated, it became clear that the important quantity is field line curvature, rather than the field strength, and the energy driving the convection comes from the work done by tension in the field lines. The combination of the tension in the field lines and their curvature produces a force. Thus fluid displacement directed along the curvature vector can tap magnetic energy. Convection cells containing a mixture of lines with large fluid pressure and low magnetic strength, and other lines with low fluid pressure and high field strength, can be driven by the release of magnetic energy if their motion results in high tension, high field strength magnetic lines crowding into central regions where the lines are shorter. As a result of the requirement that convection cells be aligned with the field, the average of the curvature over magnetic surfaces becomes the critical quantity.

Actually, the curvature of the field lines and the field strength are related. To see this, consider the vacuum field surrounding a wire carrying a current. The total magnetic stress must be in balance, since the vacuum provides nothing to push on. The gradient of the field strength would lead to an expansion of the field away from the wire, but it is held in place by the tension in the circular field lines. More detailed considerations show that the interaction of the compression and tension stresses in vacuum fields lead to a balance, perpendicular to the field lines, between $\nabla B^2/2$

and $\boldsymbol{B}\cdot\nabla\boldsymbol{B}$. In complex geometries it turns out that the latter is more fundamental in determining plasma stability than the former.

When a scalar pressure plasma is added to a vacuum equilibrium, the forces induced by tension are balanced by a sum of the fluid and magnetic pressures, $\nabla(\mu_0 p+B^2/2)$. Thus this latter quantity is a measure of the magnetic tension forces. As a result, a distinct improvement over Eq. (1) is obtained by taking the well parameter to be

$$W = 2\frac{V}{\left\langle B^2\right\rangle}\ \frac{d}{dV}\left\langle \mu_0\, p(V) + \frac{B^2}{2}\right\rangle \quad , \tag{4}$$

where positive W is favorable for containment.

Progress from this point depended on obtaining rigorous stability criteria for simple configurations. The first simple criterion applicable for a tokamak was due to Suydam[4]. He considered a screw pinch, that might also be called a straight, circular cross section tokamak. The equilibrium for this configuration, in cylindrical (r, θ, z) coordinates, satisfies

$$\mu_0 \frac{dp}{dr} + B_z \frac{dB_z}{dr} + B_\theta \frac{dB_\theta}{dr} + \frac{B_\theta^2}{r} = 0 \quad . \tag{5}$$

Note that the last term is the equilibrium force associated with field line tension. Suydam used the radius r as the label for the magnetic surfaces and derived the criterion for stability to slowly convecting localized modes as

$$-\frac{r}{4}\left(\frac{q'}{q}\right)^2 - \frac{2\mu_0 p' r'}{B_z^2} < 0 \quad . \tag{6}$$

A factor of r', that is unity when r is the surface label, has been added to the last term so that this criterion has the same behavior as Eq. (3) under change of surface label. Here $q(r)$ is the safety factor, defined as

$$q = \frac{2\pi r B_z}{L B_\theta} \quad , \tag{7}$$

and L is the periodicity length. The physics we are looking for is hidden in the normalization of Eq. (6). To understand this result, recast Eq. (6) as

$$D_I = -\frac{1}{4} - \frac{p'V'}{q'^2}\ \frac{q^2 r'}{V'}\ \frac{2\mu_0}{rB_z^2} < 0 \quad . \tag{8}$$

This normalization of Eq. (6) yields the quantity D_I that is useful for describing the eigenfunctions of the perturbation.

It is natural to view Eq. (8) or Eq. (6) as the sum of two different causes. The first term in Eq. (8) represents stabilization by shear of the magnetic field configuration. As indicated above, normalizing this term to $-1/4$ puts the criterion in a standard form. The second term contains information about the magnetic well stabilization. There are two major factors in this second term that correspond to the two factors of Eq. (3). The first, $p' V'/q'^2$, is the normalized pressure gradient. It contains the principal information about plasma profiles in a form that is independent of the coordinate used to label magnetic surfaces. Thus the second, geometric factor can be identified with the well. Expressing this well term in a form analogous to Eq. (3) we are led to a definition of the Suydam well parameter, noting that the volume is $V = \pi r^2 L$

$$W_s = -\frac{q^2\, r'\, 2\mu_0}{V' r B_z^2} = -\frac{4\pi}{L^3} \frac{\mu_o}{B_\theta^2} \quad . \tag{9}$$

This also has a form that is independent of the choice of surface label. The significant feature of this relation is that the well depends on B_θ^2, which can be identified with the tension force in Eq. (5). This tension force leads to a negative W_s that is unfavorable for confinement. The expression of Eq. (9), obtained from a rigorous stability calculation, is to be compared with the approximate well evaluated for this configuration from Eq. (4) as given in Eq. (5.34) of reference,[2]

$$W = -\frac{B_\theta^2}{B^2} \quad , \tag{10}$$

and the approximate well evaluated for this configuration from Eq. (1),

$$\hat{W} = -\frac{1}{B^2}\left(B_\theta^2 + r\mu_0 \frac{dp}{dr}\right) \quad . \tag{11}$$

The quantity W_s of Eq. (9) seems very different from W of Eq. (10), but since they are both negative and exhibit a B_θ^2 dependence, the difference between them is a matter of normalization. Of particular interest is the distinct difference between $\hat{W}$ of Eq. (11) and W of Eq. (10). The second, apparently stabilizing term of Eq. (11) is not reflected in the more rigorous Suydam criterion. Thus Eq. (4) is a much better estimate for a magnetic well than is Eq. (1).

Another early rigorous stability criterion appeared in the energy principle paper[3]. The equilibrium can be described as a variation of a tokamak with vanishing toroidal magnetic field. Then the magnetic field lines are short and closed, and this has a significant effect on the way slow sound modes are treated. To compare their results to those of this paper it is necessary to eliminate the slow sound wave by setting $\gamma p = 0$. In their paper the average field line curvature is contained in a quantity $V''(\chi)$ where the surface label χ is the poloidal flux. Intuitively, a positive value of V'' implies that the contained volume increases faster than the flux, *i.e.*, the field weakens in the outer regions. This is unfavorable for confinement so that $-V''$ is a candidate for a well parameter. The tradition of using V'' is followed and generalized in this note. In particular, Eq. (6.27) of Ref. 3 and Eq. (21) of this review agree in the appropriate limits.

The result, Eq. (8), of Suydam[4] was soon generalized to arbitrary toroidal configurations[5,6]. One form[7] of the generalization of the stability criterion of Eq. (8) is

$$D_I = -\frac{1}{4} + \frac{p'V'\chi'^4}{\left(\psi'\chi'' - \chi'\psi''\right)^2} W_M < 0 \qquad , \tag{12}$$

where

$$\begin{aligned} W_M = &-\frac{\mu_0}{\chi'^2 p'}\left\langle \frac{B^2}{|\nabla\chi|^2}\left[p'V'' + (\sigma\chi' - J')\psi'' - (\sigma\psi' - I')\chi''\right]\right\rangle \\ &+ \frac{\mu_0^2 p'V'}{\chi'^2}\left\langle\frac{1}{B^2}\right\rangle\left\langle\frac{B^2}{|\nabla\chi|^2}\right\rangle + \frac{\mu_0^2 V'}{p'}\left[\left\langle\frac{\sigma^2 B^2}{|\nabla\chi|^2}\right\rangle\left\langle\frac{B^2}{|\nabla\chi|^2}\right\rangle - \left\langle\frac{\sigma B^2}{|\nabla\chi|^2}\right\rangle^2\right] \qquad , \end{aligned} \tag{13}$$

can be identified as the Mercier form of the well parameter. Again, positive W_M is favorable for containment. Here ψ and χ are the toroidal and poloidal magnetic fluxes, I and J are the toroidal and poloidal current fluxes, and σ is the parallel current, $\boldsymbol{j}\cdot B/B^2$.

This can be put in a better form. First, introduce the safety factor $q = \psi'/\chi'$, and use it to eliminate ψ,

$$\psi' = q\chi' \qquad . \tag{14}$$

Further we use the MHD equilibrium identity[8]

$$p'V' - J'\psi' + I'\chi' = 0 \qquad , \tag{15}$$

to eliminate I'. Then the denominator of Eq. (12) becomes

$$\left(\psi'\chi'' - \chi'\psi''\right)^2 = q'^2\chi'^4 \qquad , \tag{16}$$

and a factor of the first term of Eq. (13) becomes

$$p'V'' + (\sigma\chi' - J')\psi'' - (\sigma\psi' - I')\chi'' = p'\chi'\left(\frac{V'}{\chi'}\right)' + (\sigma\chi' - J')q'\chi' \qquad . \tag{17}$$

From Eq. (16), the coefficient of W_M in Eq. (12) is the same as the coefficient of W_s in Eq. (8), and thus the Mercier well W_M is a generalization of the Suydam well.

One thing remains to demonstrate that W_M is useful, and that is to show that none of the terms in Eq. (13) actually depend inversely on the pressure gradient. This will be done for the axisymmetric case, which is the only case of real interest. It is useful to start by defining a surface function f as

$$f = \frac{q\chi'}{V'} \frac{2\pi}{\left\langle 1/R^2 \right\rangle} \quad , \tag{18}$$

which is in fact RB_T, where R is the distance from the major axis and B_T is the toroidal field strength. In terms of f, the currents are

$$\begin{aligned} J &= -2\pi f / \mu_0 \quad , \\ \sigma &= -\frac{2\pi}{\mu_0 \chi'} \left[\frac{f \mu_0 p'}{B^2} + f' \right] \\ &= -\frac{p' V'}{q \chi'^2 B^2} \left\langle B_T^2 \right\rangle - \frac{2\pi f'}{\mu_0 \chi'} \quad , \end{aligned} \tag{19}$$

and the magnetic field strength is given by

$$B^2 = \frac{4\pi^2 f^2 + |\nabla\chi|^2}{4\pi^2 R^2} \quad . \tag{20}$$

When Eq. (19) is inserted into Eq. (13) it is straightforward to show that the terms containing f' all cancel. As a result W_M can be expressed as

$$\begin{aligned} W_M = &-\frac{\mu_0}{\chi'^2} \left\langle \frac{B^2}{|\nabla\chi|^2} \left[\chi' \left(\frac{V'}{\chi'} \right)' - \frac{q' V'}{q B^2} \left\langle B_T^2 \right\rangle \right] \right\rangle + \frac{\mu_0^2 p' V'}{\chi'^2} \left\langle \frac{1}{B^2} \right\rangle \left\langle \frac{B^2}{|\nabla\chi|^2} \right\rangle \\ &+ \frac{\mu_0^2 p' V'^3}{q^2 \chi'^4} \left\langle B_T^2 \right\rangle^2 \left[\left\langle \frac{1}{|\nabla\chi|^2 B^2} \right\rangle \left\langle \frac{B^2}{|\nabla\chi|^2} \right\rangle - \left\langle \frac{1}{|\nabla\chi|^2} \right\rangle^2 \right] \quad , \end{aligned} \tag{21}$$

which is manifestly finite in the limit that p' vanishes.

One of the interesting and useful features of this well parameter is that it exhibits a clean separation between the magnetic geometry, as displayed in W_M, and the plasma profile parameters as displayed in the coefficient $p' V'/q'^2$ of Eq. (12) with (16). In particular, the effect of small, short wavelength variations of the pressure and safety factor were studied by Greene and Chance[9]. They showed that when the equilibrium response to these variations was accounted for, W_M is unaffected by such perturbations. Thus it can be treated as a geometric quantity rather than a profile quantity.

Up to this point we have treated the shear stabilization and the magnetic well as independent, so the well can be evaluated by adding 1/4 to the Mercier stability criterion. In fact convection can be achieved in the presence of magnetic shear only if permitted by resistivity. When a calculation is actually carried out, for magnetic field aligned interchanges including resistivity in the equations[9], small differences arise. It turns out that convective instabilities depend on the sign of a quantity D_R, where

$$D_R = D_I + \left(\frac{1}{2} - H\right)^2 \quad , \tag{22}$$

A value of $D_R < 0$ predicts stable stratification. It seems preferable to identify shear stabilization with $(1/2 - H)^2$ and well stabilization with $-D_R$. Here H is a peculiar quantity defined as[10]

$$H = \frac{\mu_0 p' V'^2}{q\, q' \chi'^2} \left\langle \frac{B^2}{|\nabla\chi|^2} \right\rangle \frac{\left\langle B_T^2 \right\rangle}{\left\langle B^2 \right\rangle} \left(\frac{\left\langle 1 \big/ |\nabla\chi|^2 \right\rangle \left\langle B^2 \right\rangle}{\left\langle B^2 \big/ |\nabla\chi|^2 \right\rangle} - 1 \right) \quad , \tag{23}$$

where $\sigma = \boldsymbol{j} \cdot \boldsymbol{B}/B^2$ is the parallel current, B_T is the toroidal magnetic field, and Eqs. (18–20) have been used.

Rather than define a resistive well in the same way as the Mercier well, we make one further transformation to produce a particularly convenient form. Thus we write the well in terms of a new quantity, $V^{\dagger\dagger}$

$$\begin{aligned} V^{\dagger\dagger} &= -\frac{q'^2 \chi'^2}{\mu_o p' V'^3} \frac{1}{\left\langle B^2 \big/ |\nabla\chi|^2 \right\rangle} D_R \\ &= -\frac{q'^2 \chi'^2}{\mu_o p' V'^3} \frac{1}{\left\langle B^2 \big/ |\nabla\chi|^2 \right\rangle} \left(\frac{p' V'}{q'^2} W_M - H + H^2 \right) , \end{aligned} \tag{24}$$

so that

$$\begin{aligned} V^{\dagger\dagger} &= \frac{1}{V'^2} \left[\chi' \left(\frac{V'}{\chi'} \right)' - \frac{q'\, V'}{q} \frac{\left\langle B_T^2 \right\rangle}{\left\langle B^2 \right\rangle} - \frac{\mu_0 p' V'}{\left\langle B^2 \right\rangle^2} \left(\left\langle B^2 \right\rangle + \left\langle \frac{\left(B^2 - \left\langle B^2 \right\rangle \right)^2}{B_p^2} \right\rangle \right) \right] \\ &= \frac{1}{V'^2} \left[\psi' \left(\frac{V'}{\psi'} \right)' + \frac{q'\, V'}{q} \frac{\left\langle B_p^2 \right\rangle}{\left\langle B^2 \right\rangle} - \frac{\mu_0 p' V'}{\left\langle B^2 \right\rangle^2} \left(\left\langle B^2 \right\rangle + \left\langle \frac{\left(B^2 - \left\langle B^2 \right\rangle \right)^2}{B_p^2} \right\rangle \right) \right] \quad , \end{aligned} \tag{25}$$

B_p and B_T are the poloidal and toroidal magnetic fields as given in Eq. (20), and Eq. (18) has been used. The notation is designed to show that it is related to the V'' that occurs in the first term of Eq. (25). A negative value of $V^{\dagger\dagger}$ is favorable for confinement.

The quantity $V^{\dagger\dagger}$ is a superior form for a magnetic well parameter. It is better grounded than Eq. (21), since it comes directly from a stability criterion. It is independent of the labeling of the magnetic surfaces. As will be seen below, it is finite at the magnetic axis. This is desirable because a parameter that necessarily vanishes at a particular point carries no information about the profiles near that point. It is further satisfying that it is a rather compact formula.

Equation (24) shows $V^{\dagger\dagger}$ as a sum of the three terms. The first term, proportional to W_M depends on shape and only weakly on plasma profiles, as mentioned above. The second, proportional to H, depends linearly on the local magnetic shear, q'. In normal tokamaks it is stabilizing, but can be an important destabilizing term when the shear is reversed. The third term, proportional to H^2, depends on the local pressure gradient, and is always destabilizing.

A straight tokamak with its pressure maximum on the magnetic axis, and H ignorable, is always unstably stratified. The curvature of the poloidal field is always in the wrong direction so that $V^{\dagger\dagger}$ is positive as it is in Eq. (9). However, the situation is different in a real tokamak, since the toroidal field lines are curved. They stabilize on the inboard side, and destabilize on the outboard side. Further there is an outward, Shafranov shift so that most of the volume between two flux surfaces is on the inboard side. Thus the toroidal curvature is, on the average, stabilizing. The poloidal field, and thus the poloidal destabilization, is weakened near the magnetic axis, but the inboard-outboard asymmetry and thus the toroidal stabilization is also weakened. The resulting balance can be displayed through an analytic evaluation of $V^{\dagger\dagger}$ valid at the magnetic axis[12]

$$V^{\dagger\dagger} = \frac{1}{2\pi^2 R_0^3} \, \frac{\kappa^2+1}{2\kappa} \left(\frac{1}{q^2} - \frac{2}{\kappa^2+1} + \frac{2(\kappa-1)}{\kappa^2(\kappa+1)} \beta_p - \frac{\kappa^2-1}{\kappa^2+1} \frac{R_0}{R_T} \right) \quad . \tag{26}$$

where negative $V^{\dagger\dagger}$ is favorable for containment. Here R_0 is the major radius of the magnetic axis and κ is the elongation of the elliptical magnetic surfaces. Before examining this result, the precise definitions of the other coefficients will be discussed.

Since the stability condition is local to magnetic surfaces, it is useful to define a local poloidal beta, β_p. Thus in Eq. (26) β_p is the surface averaged inverse of the fraction of the plasma confinement generated by the poloidal magnetic field. That is, in Eq. (15) the term $J'\psi'$ is a measure of the confinement by the toroidal field, and $-I'\chi'$ is a measure of the confinement by the poloidal field. Thus the following definition has been introduced,

$$\begin{aligned} \beta_p &= -\frac{p'V'}{I'\chi'} = \frac{p'V'}{p'V' - J'\psi'} \\ &= \frac{\mu_0 p'}{\mu_0 p' + ff'\left\langle 1/R^2 \right\rangle} \quad , \end{aligned} \tag{27}$$

where Eqs. (14), (18) and (19) have been used. When $\beta_p = 1$ the plasma confinement is produced entirely by the poloidal field and when β_p is very large, plasma is almost entirely contained by the toroidal field. In configurations with small β_p the poloidal field must contain toroidal field as well as plasma pressure.

In Eq. (26) R_0/R_T is a measure of the triangularity of the magnetic surfaces. The usual triangularity parameters are unsatisfactory for various reasons, so the following alternative has been proposed[13]. Consider the axisymmetric flux surfaces, $\chi(R,z)$ and plot the curve

$$\partial\chi / \partial R = 0 \quad , \tag{28}$$

as shown for a particular case in Fig. 1. This will be called the tip curve. Then compare the radius of curvature of the tip curve near the magnetic axis, R_T, with the major radius of the magnetic axis, R_0, to find the ratio R_0/R_T. Clearly, small values of R_T imply large triangularity. For an example, consider the following. Introduce a coordinate $x = R - R_0$ so that the tip curve is given approximately, for small x, by

$$2 x_t R_T + z^2 = 0 \quad , \tag{29}$$

and the ratio of the coefficient of x to the coefficient of z^2 is the desired $2R_T$. Then if the elongation of the elliptical surfaces is $\kappa = b/a$, the inverse aspect ratio of the configuration is $\varepsilon = a/R_0$, and a triangularity parameter is defined as $\delta = -x/a$, we find

$$\frac{R_0}{R_T} = \frac{2\delta}{\kappa^2 \varepsilon} \quad . \tag{30}$$

Thus R_0/R_T is a reasonable measure of triangularity, and, from the form of Eq. (26) a direct measure of an important effect of shaping on convective instabilities.

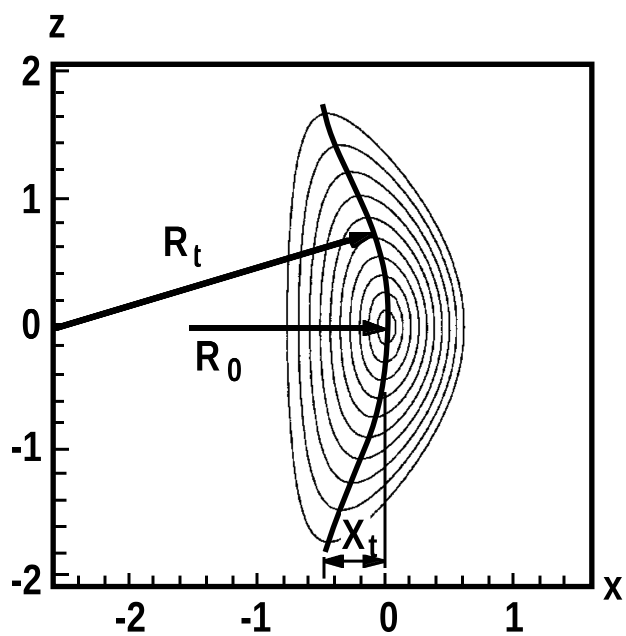

Fig. 1. The relation of the tip curve of Eq. (28) and the magnetic flux surfaces is shown for an illustrative configuration. The radius of curvature of the tip curve R_T, the radius of the magnetic axis R_0, and the concavity of the tip curve x_t are shown.

Now turn to a detailed examination of the terms in Eq. (26). The first of the bracketed terms arises from the poloidal field curvature, while the other three depend on the toroidal field. In the limit of small safety factor the poloidal field is relatively strong and its destabilization dominates in Eq. (26). For larger values of the safety factor the curvature of the toroidal component of the field dominates, so that normal stratification is stabilized. For plasma configurations with a circular cross section only the first of the toroidal terms is nonvanishing and the critical safety factor for stability is $q = 1$. For increasing values of the elongation κ the toroidal stabilization is reduced. In the limit, for low plasma pressure, unshaped configurations, and $\beta_p = R_0/R_T = 0$, the critical q is approximately $q = \kappa/\sqrt{2}$. The last of the three toroidal terms measures the stabilization achieved from favorable triangularity. It can be seen from Fig. 1 that the magnetic surfaces are spread out near the inboard corners, so that these regions where the toroidal curvature is favorable are more heavily weighted. This is the origin of the influence of triangularity. Near the magnetic axis this effect is much reduced, but it is competitive with the effects of the reduced poloidal field. The curvature of the tip curve puts this balance in a visible form. Finally, a principal consequence of larger plasma pressure is its effect on the plasma shape that is accounted for in the triangularity term. However, there is some left over that appears explicitly as the second of the three toroidal terms.

Equation (26) scopes out the principal features of stable stratification. A primary control is the value of the safety factor, that determines the relative strength of the poloidal and toroidal terms. Although elongation of the flux surfaces is destabilizing, it can be easily countered by appropriate shaping that heavily weights the inboard side in the flux surface averages. On the other hand, Eq. (26) misses many details and is not a substitute for numerical evaluation of Eq. (25).

In Fig. 2 a typical plot of the $V^{\dagger\dagger}$ profile is shown. In this configuration the value of the safety factor at the axis was 0.95, so the $V^{\dagger\dagger}$ stability criterion was violated. Away from the axis the stabilizing effect of toroidal curvature came to dominate, so that only the core is convectively unstable.

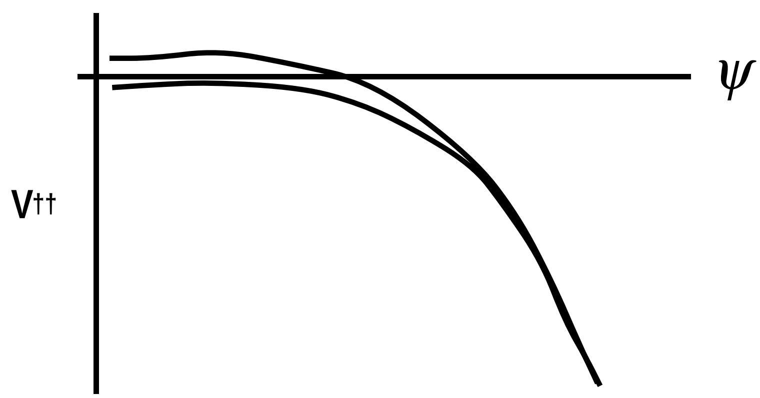

Fig. 2. The profile of $V^{\dagger\dagger}$ for two illustrative configurations with large triangularity, one with safety factor less than unity near the magnetic axis, and the other with safety factor greater than unity everywhere. The quantity $V^{\dagger\dagger}$ is positive and destabilizing near the axis, in the configuration with smaller safety factor, and stabilizing everywhere for the other configuration.

A most important question is, what would these considerations predict about tokamaks? It is useful to return to the atmosphere for guidance. A primary observation about convection in a solar heated atmosphere is that it is gusty. The conclusion to be drawn from this is that convection must be inefficient on small scales, so that it is possible to accumulate enough energy to make a gust. A second thought is that the gusts might be able to penetrate and mix into a mildly stable upper layer, so that the degree of stability could be important when there is a significant stability profile. However, this depends on the size of the gusts, and so is outside of the ideas of this review. Returning to tokamaks, it is reasonable to conclude that the plasma pressure gradient should be small or vanishing in regions where $V^{\dagger\dagger}$ is positive, but could be large where it is negative. The significance of the magnitude of $V^{\dagger\dagger}$ is uncertain in the absence of information on the gustiness of the convection.

The conclusion of this exercise is that $V^{\dagger\dagger}$ of Eq. (25) is the best indicator of stable stratification in a tokamak. If this quantity is negative, the usual tokamak with radially decreasing pressure is stably stratified. In general, stable stratification can be obtained if the safety factor is sufficiently large. Stable stratification is most diffucult to achieve in tokamaks with elongated cross section that are insufficiently D–shaped.

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References

1. M. N. Rosenbluth and C. L. Longmire, Ann. Phys. (NY) **1**, 120 (1957).
2. J. P. Freidberg, *Ideal Magneto-hydro-dynamics* (Plenum, New York, 1987).
3. I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. (London) **A244**, 17 (2958).
4. B. R. Suydam, *Proc. U. N. Conf. Peaceful uses At. Energy, 2nd, Geneva, 1958*, (Columbia Univ. Press, New York, 1959) Vol. 31, p. 85.
5. C. Mercier, Nucl Fusion Suppl. Pt. 2, 801 (1962).
6. J. M. Greene and J. L. Johnson, Phys. Fluids **5**, 510 (1962).
7. J. M. Greene and J. L. Johnson, Plasma Phys. **10**, 729 (1968).
8. M. D. Kruskal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958).
9. J. M. Greene and M. S. Chance, Nucl. Fusion **21**, 453 (1981).
10. A. H. Glasser, J. M. Greene, and J. L. Johnson, Phys. Fluids **18**, 875 (1975).
11. R. L. Dewar, R. C. Grimm, J. L. Johnson, E. A. Frieman, J. M. Greene, and P. H. Rutherford, Physics of Fluids **17**, 930 (1974).
12. K. E. Weimer, E. A. Frieman, and J. L. Johnson, Plasma Phys. **17**, 645 (1975).
13. F. J. Helton, J. M. Greene, T. Ohkawa, P. A. Politzer, and A. D. Turnbull, Nucl. Fusion **31**, 487 (1991).