A MULTI-GRID SOLVER FOR UP-DOWN ASYMMETRIC TOKAMAK EQUILIBRIA*

S.K.Wong, M.S. Chu, Y.R.Lin-Liu

General Atomics
San Diego, California

The computer code TOQ produces tokamak equilibria in a coordinate system which uses the poloidal flux surfaces as coordinate surfaces. It is well-suited for providing accurate equilibria to stability analyses. It has been extremely useful for beta-optimization in modeling Advanced Tokamak scenarios and the study of bootstrap current driven Spherical Torus. To solve the Grad-Shafranov equation, TOQ uses a mult-grid (MG) package[1], which has been shown to be very robust in solving elliptic problems. The MG algorithm as implemented has to be modified in order to deal with up-down asymmetric equilibria. In this work we discuss an implementation of the new MG algorithm and give examples of up-down asymmetric high-beta equilibria using DIII-D geometry and the ITER geometry. We also explore the stability of such equilibria vis a vis the up-down symmetric ones.

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1. **P.M. De Zeeuw, Matrix-dependent prolongations and restrictions in a black box multigrid solver, J. Comput. and Appl. Math. 33 (1990) 1-27**

OBJECTIVES

Extend the MHD equilibrium code (TOQ) to

- **solve for fixed-boundary equilibria with boundary flux surfaces which can be up-down asymmetric**

- **employ multi-grid method for 2-D elliptic equations with periodic boundary condition in one dimension for numerical efficiency**

BACKGROUND

- For up-down symmetric equilibria, DeLucia, Jardin and Todd[1] obtained solutions of finite-difference Grad-Shafranov equation in flux coordinates via iterative change of coordinates

- A modified approach at GA (Miller and Lin-Liu) was implemented in TOQ

- Recent use of a multi-grid method package[2] in TOQ improved speed of convergence by a large measure, allowing exploration of new equilibrium regimes[3]

- Need for up-down asymmetric equilibria exists both for DIII-D and future tokamak reactors such as devices similar to ITER

- Periodic boundary conditions are required for general equilibria

- Neither TOQ nor the multigrid package are equipped to handle periodic boundary conditions

1. J. DeLucia, S.C. Jardin, and A.M.M. Todd, J. Comput. Phys. 37 183 (1980)
2. P.M. de Zeeuw, J. Comput. Appl. Math. 33 1 (1990)
3. R.L. Miller, Y.R. Lin-Liu, A.D. Turnbull, and V.S. Chan, L.D. Pearlstein, O. Sauter and L. Villard, Phys. Plasmas 4 1062 (1997)

DEVELOPMENTAL TASKS

1. Create iterative 2-D elliptic equation solver with periodic boundary condition in one dimension

2. Implement multi-grid approach for the above problem

3. Modify TOQ to accommodate periodic boundary conditions and incorporate multi-grid solver

4. Validate against existing TOQ

THE 2-D ELLIPTIC PROBLEM

$$A\frac{\partial^2 f}{\partial \rho^2} + B\frac{\partial^2 f}{\partial \rho \partial \theta} + C\frac{\partial^2 f}{\partial \theta^2} + D\frac{\partial f}{\partial \rho} + E\frac{\partial f}{\partial \theta} + Ff = g$$

in the domain $a \le \rho \le b \quad 0 \le \theta \le 2\pi$ **with Dirichlet boundary conditions for** ρ **and periodic conditions for** θ**. Coefficients depend on** ρ, θ

Finite-differencing leads to alternate representations:

FORM 1: NINE-POINT STENCIL

$$\begin{array}{ll} a_7 f_{i-1\,j+1} + a_8 f_{i\,j+1} + a_9 f_{i+1\,j+1} & \\ a_4 f_{i-1\,j} \;\; + a_5 f_{i\,j} \;\; + a_6 f_{i+1\,j} & = g_{i\,j} \\ a_1 f_{i-1\,j-1} + a_2 f_{i\,j-1} + a_3 f_{i+1\,j-1} & \end{array}$$

FORM 2: BLOCK TRIDIAGONAL $Lf = g$

$$L = \begin{pmatrix} D_1 & P_1 & & & \\ M_2 & D_2 & P_2 & & \\ & & . & & \\ & & & . & \\ & & & M_n & D_n \end{pmatrix}$$ **coupling** ρ **grid lines.**

$$D, P, M = \begin{pmatrix} \times & \times & . & . & \times \\ \times & \times & \times & & \\ & & . & & \\ & & & . & \\ \times & & & \times & \times \end{pmatrix}$$ **coupling** θ **values (cyclic tridiagonal)**

SOLUTION BY ILLU DECOMPOSITION

The matrix equation is iteratively solved by a modified Thomas algorithm which involves the calculation of $\overline{D}_j$ by

$$\overline{D}_1 = D$$
$$\overline{D}_j = D_j - \textit{tridiagal part of } M_j \overline{D}_{j-1}^{-1} P_{j-1} \qquad j = 2,..,n$$

which enter into the approximate LU decomposition

$$L \approx \begin{pmatrix} I & & & & \\ M_2\overline{D}_1^{-1} & I & & & \\ & M_3\overline{D}_2^{-1} & & & \\ & & \cdot & & \\ & & & M_n\overline{D}_{n-1}^{-1} & I \end{pmatrix} \begin{pmatrix} \overline{D}_1 & P_1 & & & \\ & \overline{D}_2 & P_2 & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \overline{D}_n \end{pmatrix}$$

The solution of linear equations with cyclic tridiagonal matrix C employs a decomposition

$$C = T + u\tilde{v}$$

where T is a tridiagonal matrix and u, v are column vectors, followed by Sherman-Morrison's formula

$$C^{-1} = T^{-1} - \frac{(T^{-1}u)(\tilde{v}T^{-1})}{1+\lambda} \qquad \lambda = \tilde{v}T^{-1}u$$

Thomas algorithm is used for solving tridiagonal systems.

CONVERGENCE IS ACCELERATED USING MULTIGRID METHOD

WHICH REPEATEDLY USES MGCS CYCLES[2]

1. Initial guess for f

2. form residue

$$r_1 = g - Lf$$

3. restrict r_1 **to successively coarser grids**

$$r_1 \to r_2 \to \cdots \to r_{l\,\max}$$

4. solve residue equation on coarsest grid:

$$L_{l\,\max}\delta f_{l\,\max} = r_{l\,\max}$$

5. for $l = l\max \quad to \quad 2$

prolongate δf_ℓ **to finer grid:** $\delta f_\ell \to \delta f_{\ell-1}$

smooth $\delta f_{\ell-1}$ **by relaxation on** $L_{\ell-1}\delta f_{\ell-1} = r_{\ell-1}$

6. obtain corrected solution

$$f \to f + \delta f_1$$

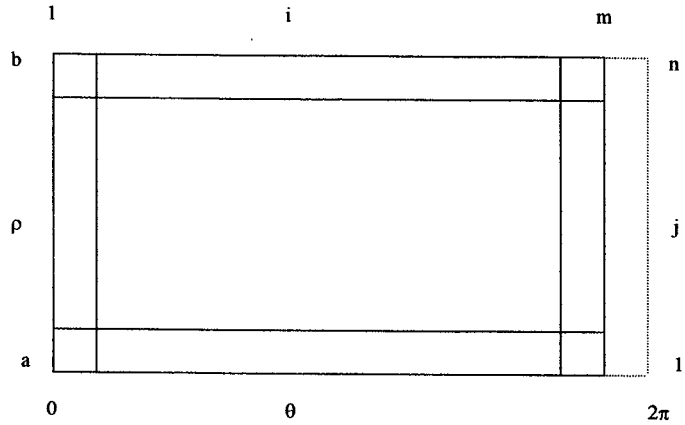
7. smooth solution f **by relaxation on** $Lf = g$

COMPONENTS IN MULTI-GRID METHOD

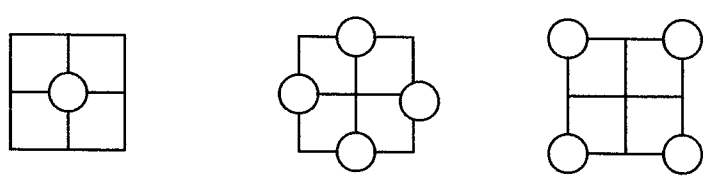
GRID LAYOUT

$$m = 2^{m'} \qquad n = 2^{n'} + 1 \qquad \Delta\theta = \frac{2\pi}{m} \quad \Delta\rho = \frac{b-a}{n-1}$$

$$m_C = m/2 \quad n_C = (n+1)/2 \quad \Delta\theta_C = \Delta\theta/2 \quad \Delta\rho_C = \Delta\rho/2$$



PROLONGATION P_ℓ **:** ○ ⟶ + **weights**



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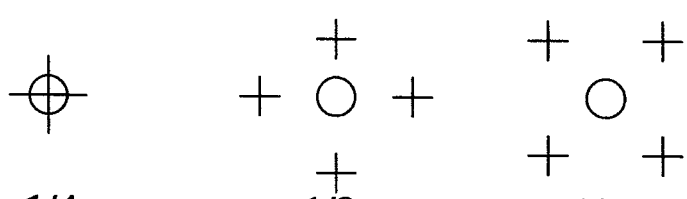
RESTRICTION R_ℓ**:** + ⟶ ○ **weights**



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RESTRICTION OF LINEAR OPERATOR: $L_\ell = R_{\ell+1} L_{\ell+1} P_\ell$

SMOOTHING: **iterate once with ILLU**

SOLVING: **iterate 8 times with ILLU**

TREATMENT OF CO-ORDINATE ORIGIN

When one boundary is the co-ordinate origin $(\rho = 0 \quad at \quad j = 1)$**, the equations there become**

$$f_{i1} = f_0 \qquad i = 1, \ldots, m \qquad\qquad f_0 = \frac{1}{m}\sum_{i=1}^{m} f_{i2}$$

which do not conform to forms 1 or 2. The problem can be solved by the following method (Lin-Liu and Miller):

1. Obtain solution to "homogeneous" problem:

$$Lf^{(0)} = 0 \qquad f^{(0)} = \begin{cases} 1 & j = 1 \\ 0 & j = n \end{cases}$$

2. Obtain solution to "inhomogeneous" problem"

$$Lf^{(1)} = g \qquad f^{(1)} = \begin{cases} 0 & j = 1 \\ given\ values & j = n \end{cases}$$

3. The required solution is

$$f = f^{(1)} + \lambda f^{(0)}$$

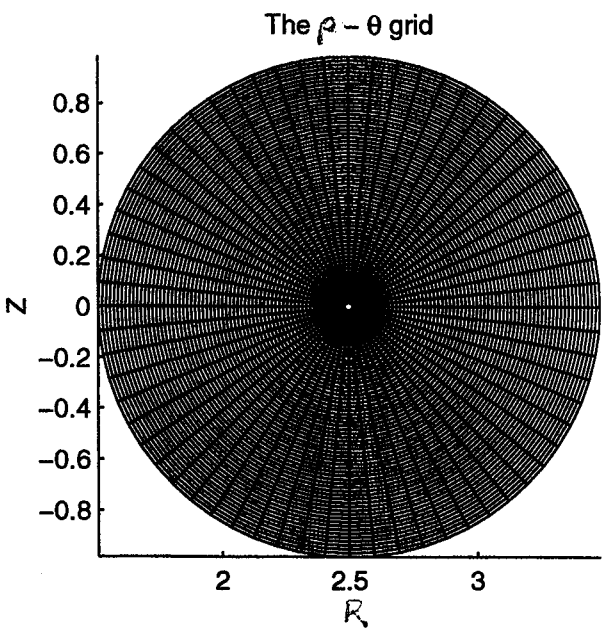
where λ **is determined by the requirement**

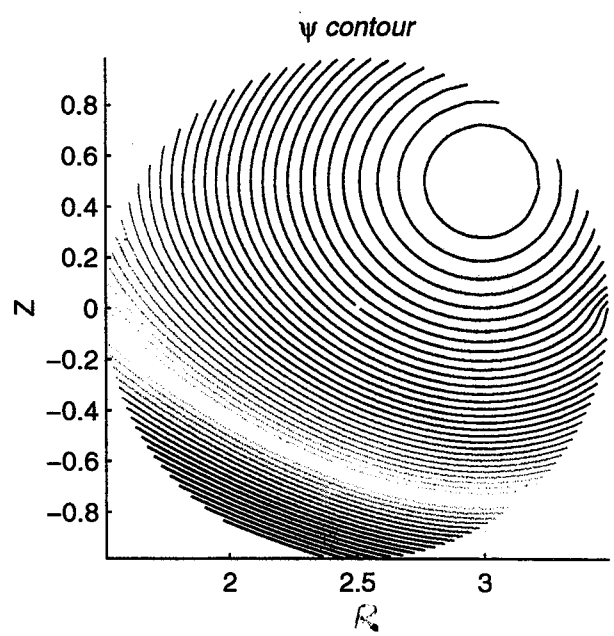
$$f_0 = \frac{1}{m}\sum_{i=1}^{m} f_{i2}$$

VALIDATION OF ELLIPTIC EQUATION SOLVER

Soloviev equilibrium with magnetic axis at (R_a, Z_a) is solved in toroidal co-ordinate system centered at (R_0, Z_0). The boundary is a circle on which the poloidal flux ψ assumes values given by the analytic equilibrium.

On a 64x64 grid, with the convergence criterion that L-2 norm of successive correction be less than 10^{-6}, the solution is obtained with point-wise accuracy of order 10^{-3}.





COMPUTATIONAL EFFICIENCY: The CPU times in seconds on the K machine for various algorithms are as follows.

grid	SOR	ILLU	Multi-grid	
			4 levels	5 levels
32x33	9.3	1.86	0.70	0.70
64x65	----	17.34	3.23	3.10
128x129	----	468.50	15.61	17.56

THE PROBLEM SOLVED BY TOQ

- **Solve Grad-Shafranov equation in general coordinate system** ρ,θ

$$\frac{R^2}{J}\left[\frac{\partial}{\partial\rho}\left(h_{\rho\rho}\frac{\partial\chi}{\partial\rho}+h_{\rho\theta}\frac{\partial\chi}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(h_{\theta\rho}\frac{\partial\chi}{\partial\rho}+h_{\theta\theta}\frac{\partial\chi}{\partial\theta}\right)\right]=-(\psi_b-\psi_a)(4\pi p'+ff')$$

where $\chi=\frac{\psi-\psi_a}{\psi_b-\psi_a}$, ψ_b **and** ψ_a **are poloidal fluxes at the boundary and the magnetic axis respectively.**

The metric tensor components are

$$h_{\rho\rho}=(J/R^2)(R_\theta^2+Z_\theta^2)\quad h_{\theta\theta}=(J/R^2)(R_\rho^2+Z_\rho^2)\quad h_{\rho\theta}=-(J/R^2)(R_\rho R_\theta+Z_\rho Z_\theta)$$

and $R(\theta,\rho),Z(\theta,\rho)$ **are cylindrical coordinates. The Jacobian is**

$J=(R_\rho Z_\theta-R_\theta Z_\rho)$. **The shape functions** $p'(\chi)$ **and** $ff'(\chi)$ **are given. The domain in the general coordinates is defined by** $0\le\rho\le1\quad 0\le\theta\le2\pi$. **The boundary curve is prescribed, as is also the boundary value** ψ_b.

- **Determine** χ, R, Z **so that**

  1. $\chi=G(\rho)$ **, a prescribed function**

  2. θ **grid points divide poloidal flux contours into segments of equal-arc length**

GRID ITERATION SCHEMES

DeLucia, Jardin and Todd:

Step 1: **Construct initial grid** $R(\theta_i,\rho_j)\ Z(\theta_i,\rho_j)$

Step 2: **Solve finite-difference modified G-S equation which assumes poloidal flux** χ **to depend only on** ρ**:**

$$\overline{\Delta}^{*}\chi \equiv \frac{R^2}{J}\left[\frac{\partial}{\partial\rho}\left(h_{\rho\rho}\frac{\partial\chi}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(h_{\rho\theta}\frac{\partial\chi}{\partial\rho}\right)\right]=-(\psi_b-\psi_a)(4\pi p'+ff')$$

Step 3: **Construct new grid** $R'(\theta_i,\rho_j)\ Z'(\theta_i,\rho_j)$ **so that constant** ρ **surfaces are** χ **contours satisfying** $\chi = G(\rho)$**, a prescribed function.**

Step 4: **Repeat Steps 2 and 3 until convergence.**

Modified approach in TOQ

Step 2: **Solve finite-difference G-S equation**

$$\Delta^{*}\chi = -(\psi_b-\psi_a)(4\pi p'+ff')$$

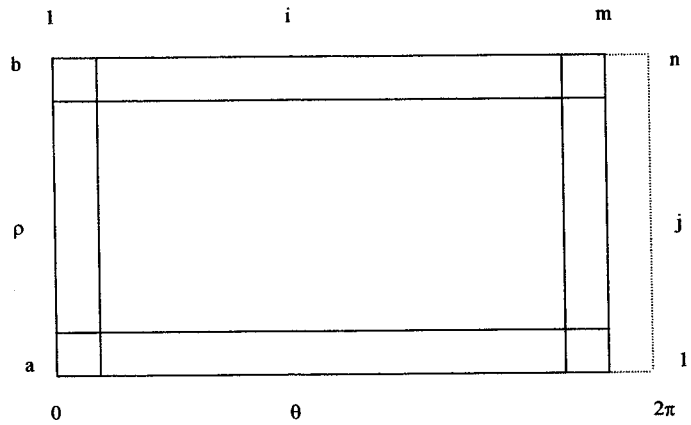
Differences:

DeLucia et al: **Solution of G-S equation is obtained only at the end of iteration together with the requisite grid**

TOQ: **Solution of G-S equation is obtained in every iteration. Purpose of iteration is to provide accurate input to stability codes.**

INITIAL CHOICE OF GRID FOR TOQ

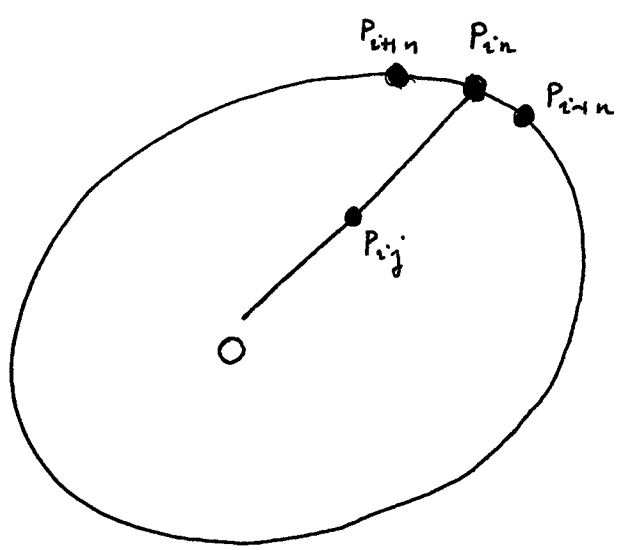
A uniform grid is chosen for (θ, ρ):



The grid points P_{ij} in physical space is chosen as follows:

1. **Uniformly spaced points P_{in} are chosen on the boundary curve**
2. **The coordinate origin O is chosen arbitrarily**
3. **Grid points P_{ij} $j = 2,..,n-1$ are laid on the line OP_{in} proportionately according to the grid function $G(\rho)$:**

$$\frac{OP_{ij}}{OP_{in}} = G(\rho)$$



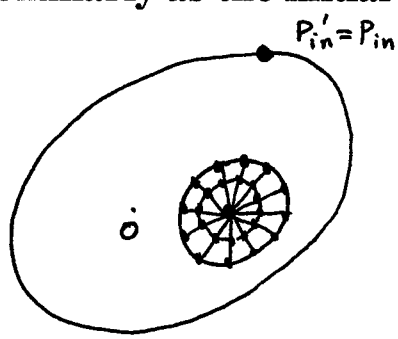
HOW NEW GRIDS ARE FORMED

1. **Start with solution χ on old grid**

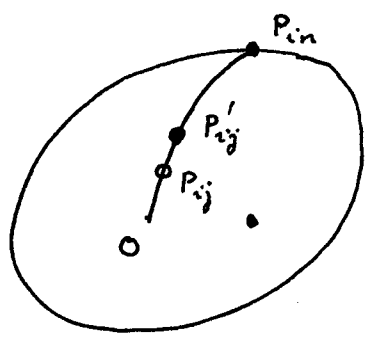
2. **Retain grid points on boundary:** $P'_{in} = P_{in} \quad i = 1,..,m$

3. **Move co-ordinate origin to location of minimum** χ**:** $O \rightarrow O'$

4. **Points** P'_{ij} **for** j **such that** $G(\rho_j) \leq \chi_0$ **where** χ_0 **is the value at the origin are laid out similarly as the initial grid:**



5. **Other points** P'_{ij} **are chosen on constant** θ **lines** OP_{in} **so that**
   Evaluation of Interpolated χ *along* OP_{in} *at* $P'_{ij} = G(\rho_j)$



6. **Points on each constant** ρ **surface are rearranged so they are equally spaced.**

WHERE TOQ USES THE 2-D ELLIPTIC EQUATION SOLVER

NONLINEAR ITERATION

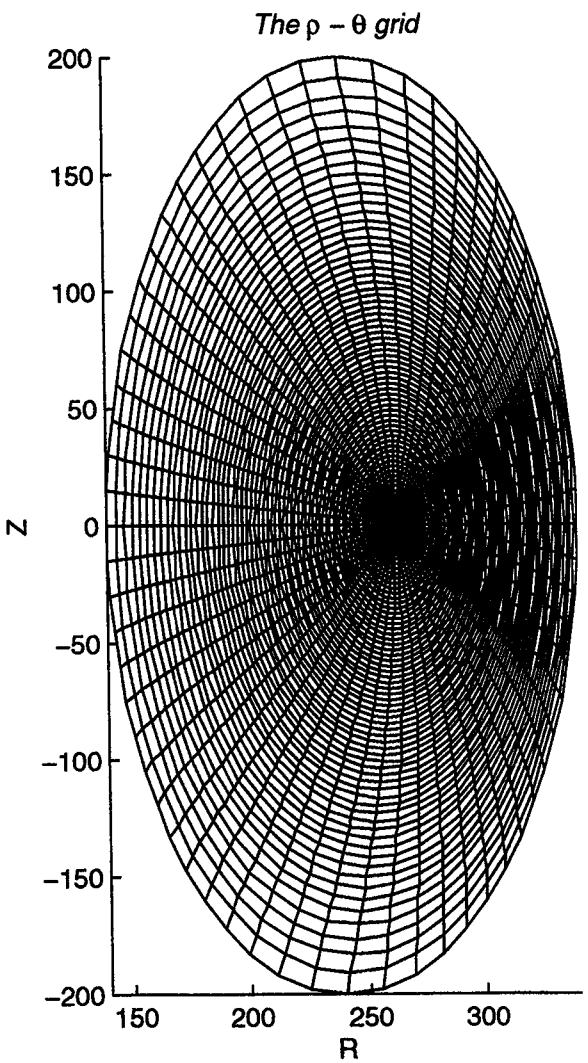
The nonlinear G-S equation is iteratively solved:

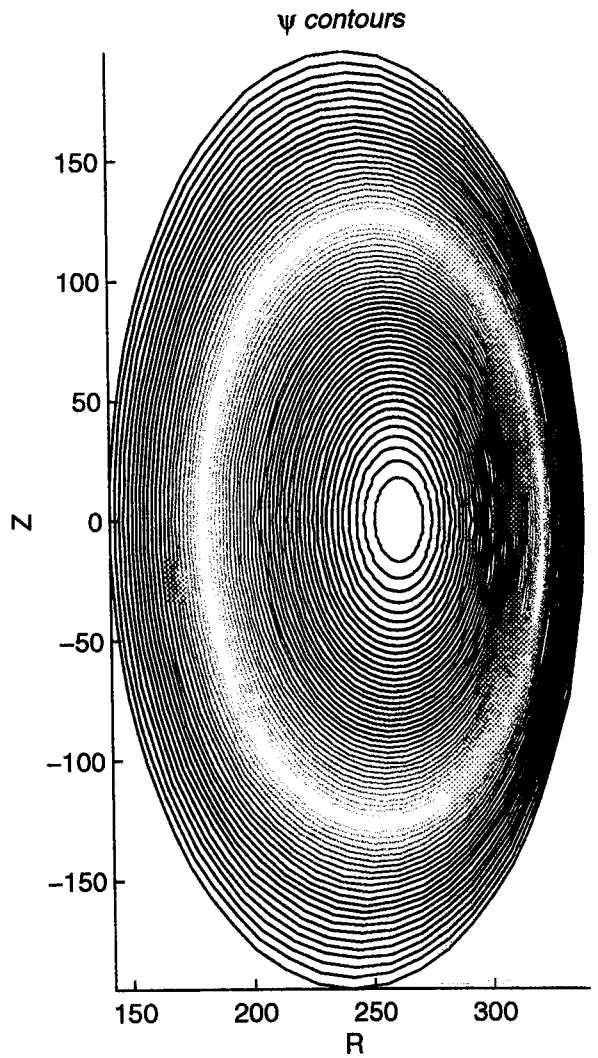
$$\Delta^* \chi^{(n+1)} = -\left(\psi_b - \psi_a^{(n)}\right)\left(4\pi p'\left(\chi^{(n)}\right) + ff'\left(\chi^{(n)}\right)\right)$$

until convergence.

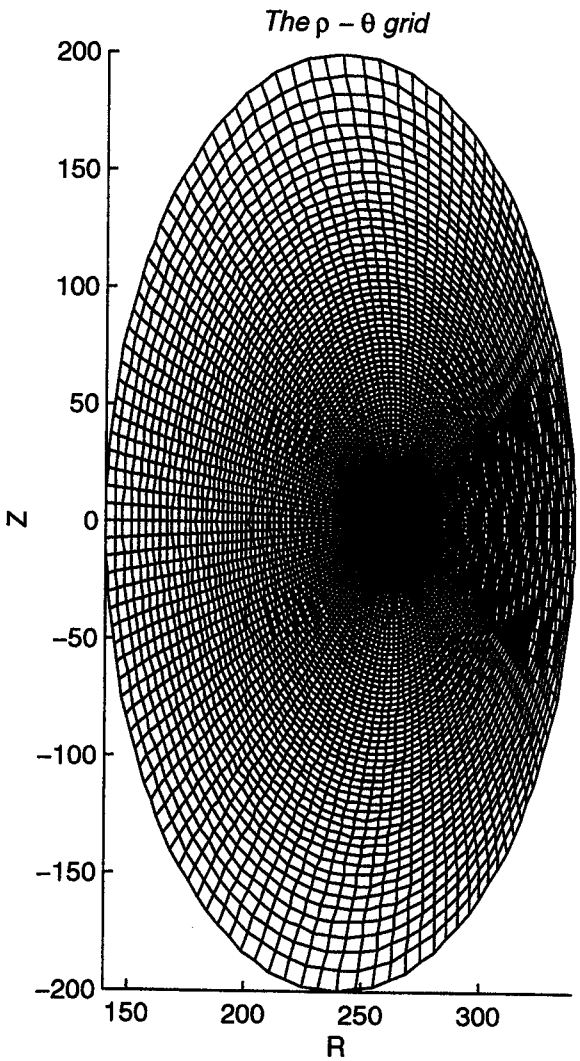
TOQ UP-DOWN SYMMETRIC EQUILIBRIUM IS REPRODUCED

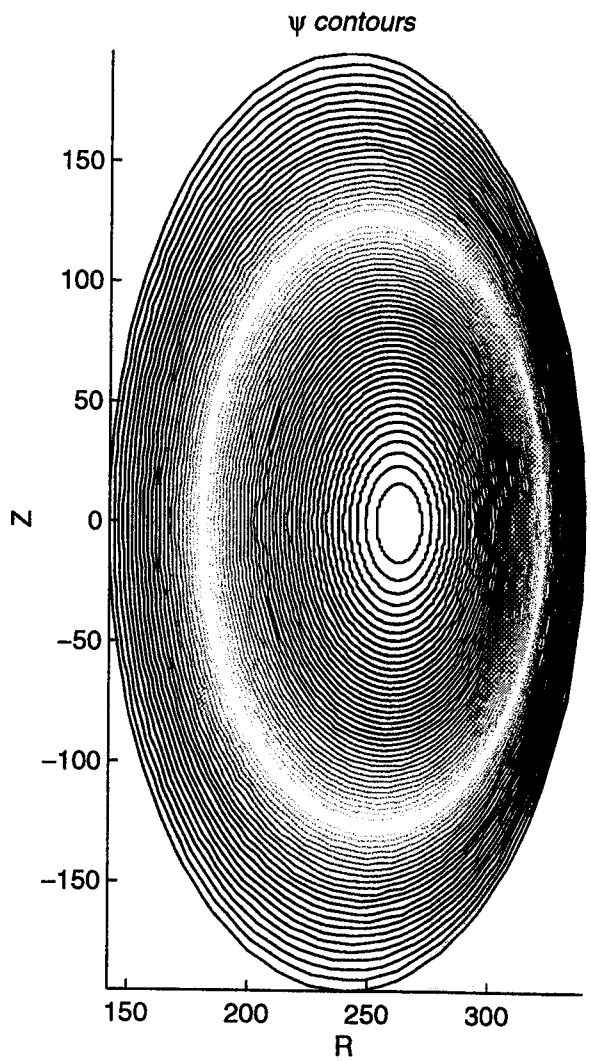
NEW CODE





EXISTING CODE



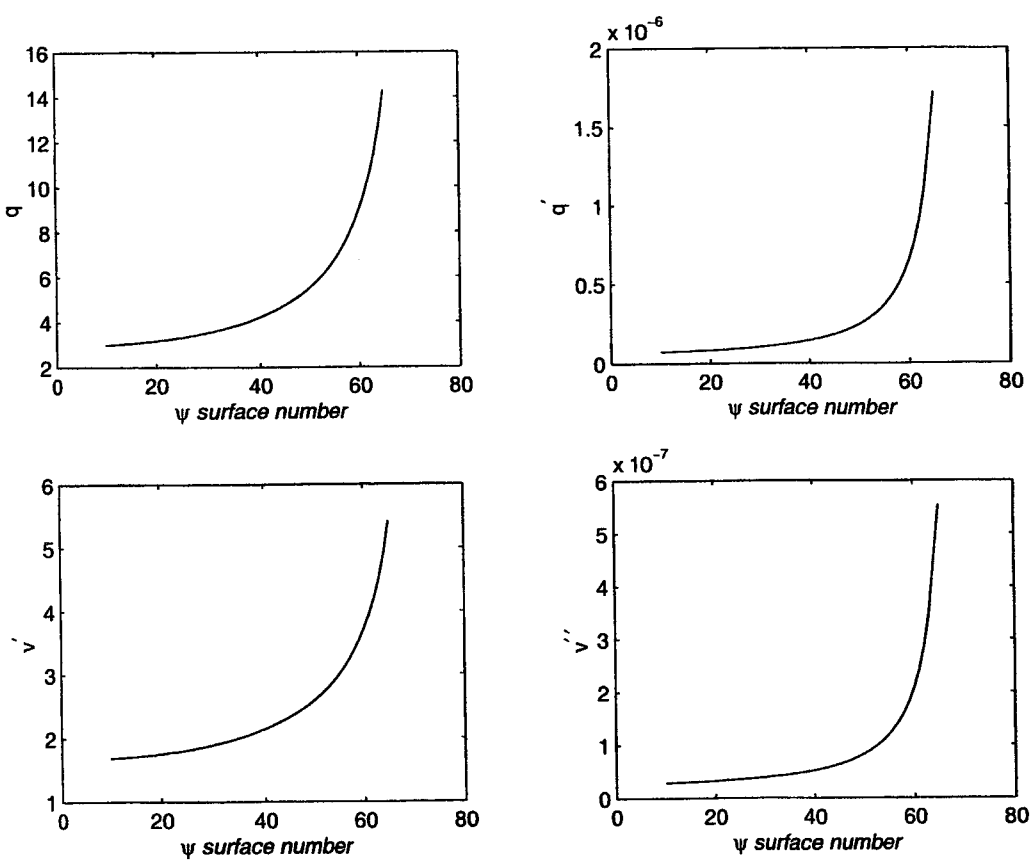


GOOD AGREEMENTS WITH EXISTING CODE

INTEGRATED QUANTITIES

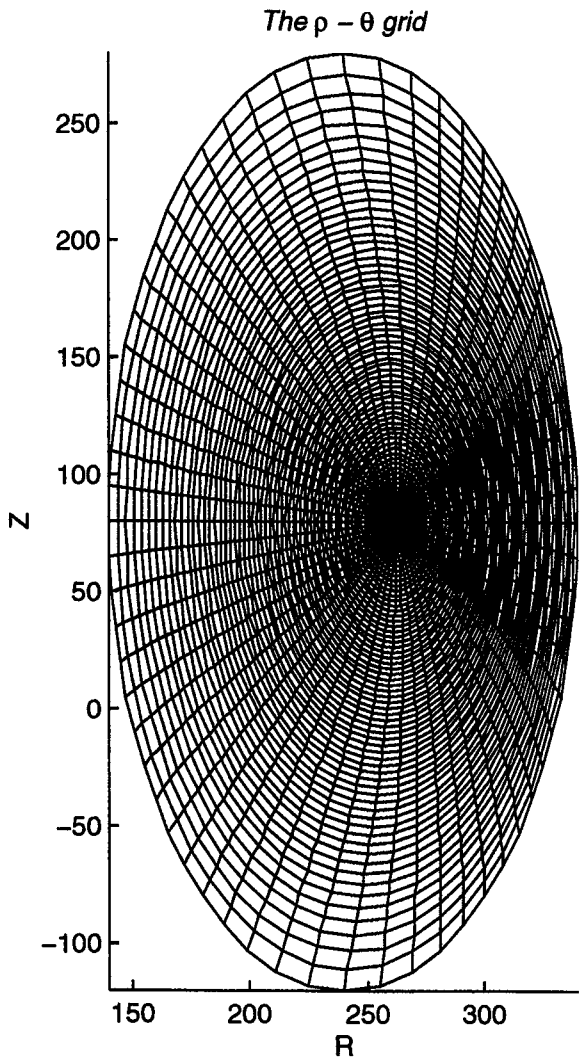
	NEW CODE	EXISTING CODE
R_a	262.493	262.474
Z_a	3.0×10^{-5}	0
$\langle\beta\rangle$	0.02313	0.02312
β_p	0.185	0.184
l_i	0.03309	0.03280
q_a	3.015	2.930
$\psi_b-\psi_a$	3.432×10^{7}	3.426×10^{7}

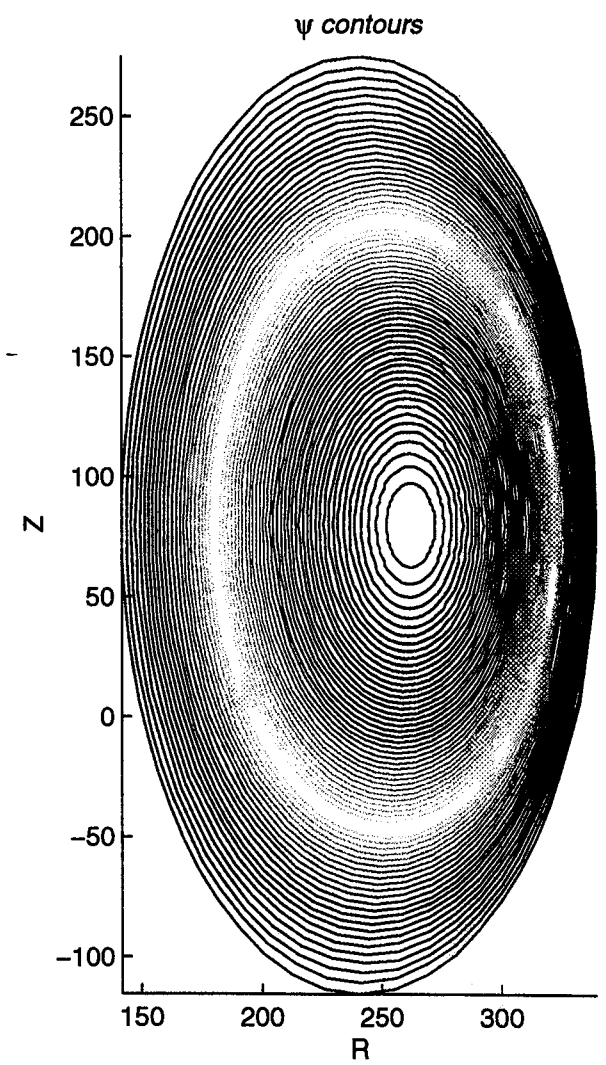
SELECTED FLUX FUNCTION PROFILES (EXISTING CODE: DASHED; NEW CODE: SOLID)



AN UP-DOWN SYMMETRIC EQUILIBRIUM SHIFTED UPWARD

When an up-down symmetric boundary is shifted upward relative to the mid-plane of the R-Z coordinates, the code is able to reproduce the intrinsically symmetric equilibrium:





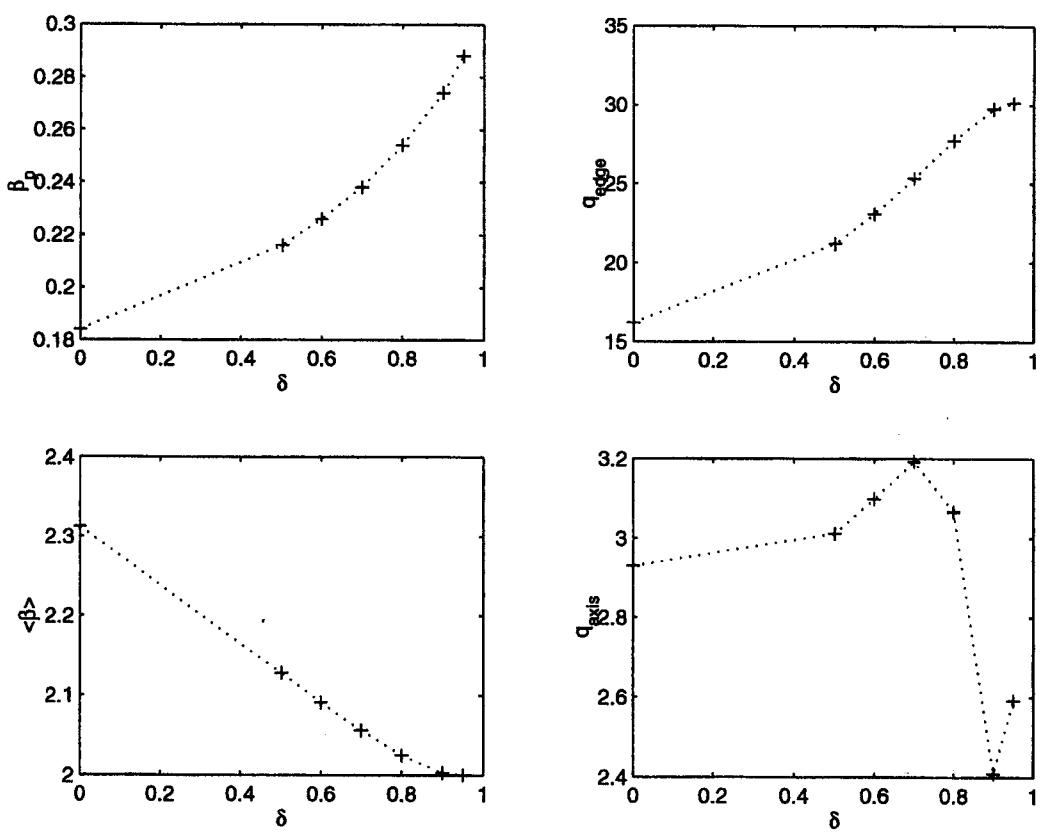
EXAMPLES OF UP-DOWN ASYMMETRIC EQUILIBRIUM

A sequence of up-down asymmetric equilibria is generated with boundary flux surfaces consisting of the upper half of an ellipse joined on to the lower part of a dee:

$$R = R_0 + R_{\max}\cos(\theta + \Delta\sin\theta) \qquad \Delta = \begin{cases} 0 & 0 \le \theta \le \pi \\ \sin^{-1}\delta & \pi \le \theta \le 2\pi \end{cases}$$

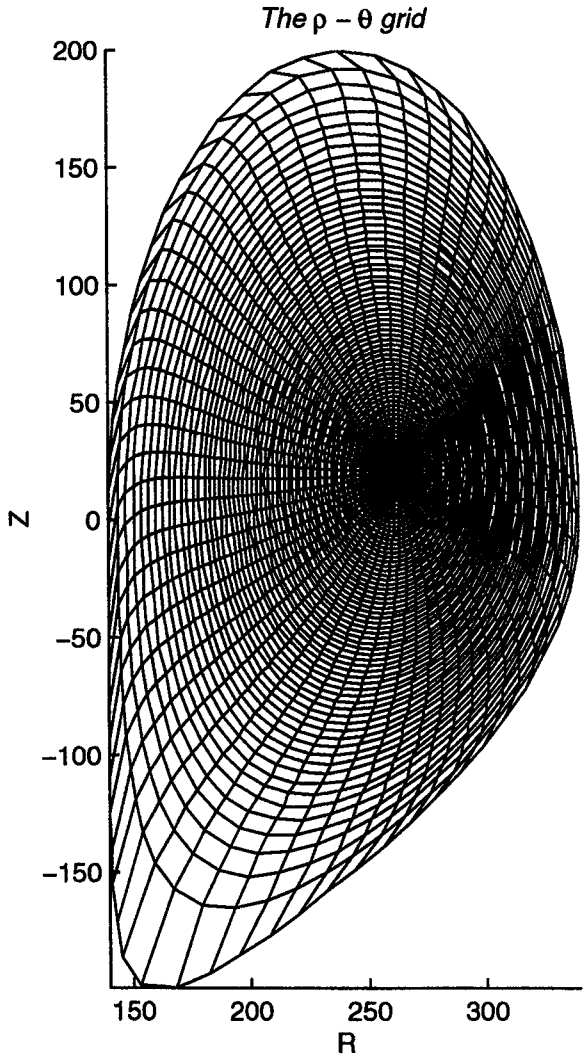
$$Z = eR_{\max}\sin\theta$$

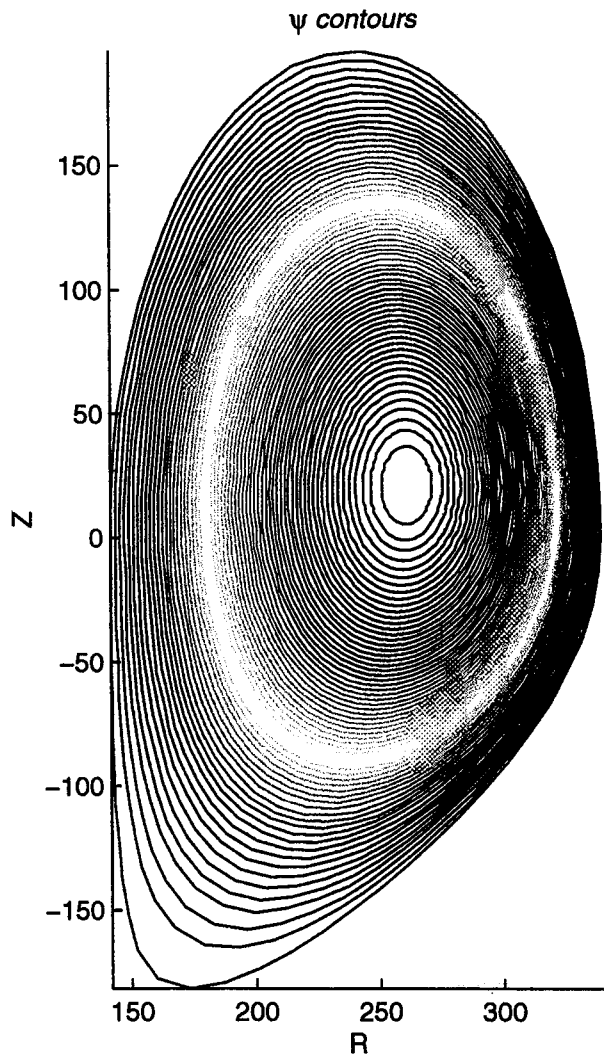
TRENDS OF SELECTED QUANTITIES



AN UP-DOWN ASYMMETRIC EQUILIBRIUM

GRIDS AND POLOIDAL FLUX CONTOURS (δ=0.8, e=2.0)





CONCLUDING REMARKS

- A multigrid solver of 2-D elliptic equations with periodic boundary condition in one dimension is developed

- TOQ is generalized to also produce up-down asymmetric equilibria

- Convergence of coordinate iteration scheme is demonstrated in TOQ for up-down asymmetric equilibria

- Computation of equilibrium quantities near magnetic axis needs refinement

- Coupling to the stability code GATO is to be performed