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Algorithms for Finding 2D MHD Equilibria with Given  
“Almost Ideal” MHD Constraints
T.H. JENSEN, General Atomics — Under the constraints of ideal MHD, the flux surface topology cannot change. This not true under the constraints of almost ideal MHD (AIMHD). Equilibrium algorithms observing AIMHD constraints can therefore be used for the study of nonlinear properties of tearing modes. Only the simplest cases are considered for which (i) ∂/∂z = 0; (ii) ∇p = 0; (iii) (B × ẑ)²/(B · ẑ)² ≪ 1. It is the aim of this work to find algorithms which can deal with cases for which the current density may be different functions of the flux function on the two sides of a singular surface so that a finite gradient of the current density at the singular surface may exist. For such cases, island formation may result in irreversible, quantifiable changes of the AIMHD constraints and a nonlinear instability of tearing modes may exist. Several approaches to making suitable algorithms for this purpose will be discussed.

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Prefer Oral Session

Prefer Poster Session

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OBJECT OF WORK:

Find means for studying nonlinear properties of tearing modes. (Ultimately to be able to handle cases with a finite current density gradient at the singular surface)

FRAMEWORK:

Single fluid MHD; \( \nabla \rho = 0 \)

"Initial equilibrium": \( \frac{\partial}{\partial t} = \frac{\partial}{\partial y} = 0 \)

"Final (forced) equilibrium": \( \frac{\partial}{\partial t} = 0 \); periodic in \( y \)

\( B = B_e \hat{z} + \psi \psi_k \hat{z} \)

TOKAMAK ORDERING: \( \frac{\psi \psi_k}{B_e} \ll 1 \)

RESULTS:

i) Means found

ii) Stability limit agrees with standard linear theory

iii) Saturated island widths are sensitive to details

BASIC CONCEPTS:

(2) An MHD equilibrium may be determined by specifying $B_2(y)$ and boundary conditions through

$$\nabla^2 y + \frac{1}{2} \frac{d}{dy} B_2(y) = 0$$

If for a real plasma external currents are added (or boundary conditions changed) $B_2(y)$ will change.

(3) An MHD equilibrium may instead be determined by specifying an ideal MHD conserved function (e.g. $\mathbf{q}(y)$).

If for a real plasma external currents are added (or boundary conditions changed), this conserved (invariant) function will not change.

(4) The response of a plasma equilibrium (given by conserved function) to external currents has information on stability and direction of dynamic development.

No information on timescale of dynamics.

(5) Ideal MHD constraints cannot be used for the study of tearing modes since change of flux function topology is impossible under such constraints.
Almost ideal MHD employs a finite number of conserved quantities (instead of a conserved function (ideal MHD)). Almost ideal MHD allows topological changes of fluxfunction and can therefore be used for studying tearing modes.

Specifics:

Geometry:

Period in $y$: 2b

CONSERVED QUANTITIES:

\[ K_0 = \int_{\text{int}} g_0(\psi(x,y)) \, dx \, dy \quad ; \quad \nu = 1, 2, \ldots \]

\( \psi_{\text{int}} \) is initial equilibrium; \( g_\nu(\psi) \) is a set of functions to be chosen.

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THE FINAL (FORCED) EQUILIBRIUM MUST SATISFY

\[ \nabla^2 \psi + j(\psi) = 0 \]

AND BOUNDARY CONDITIONS. A FUNCTION \( j(\psi) \) MUST BE FOUND SO THAT:

\[ K_\nu = \int_{\text{int}} g_\nu(\psi(x,y)) \, dx \, dy = K_0 \]

AN EXPANSION AROUND THE FINAL SOLUTION SHOWS THAT IF

\[ j(\psi) = \sum_{\nu} \alpha_\nu \frac{\partial g_\nu(\psi)}{\partial \psi} \]

THEN THE MATRIX, \( M_{\psi,\psi} \), RELATING THE \( \delta \psi \)'s AND THE \( \delta K_\nu \)'s \( (\delta K_\nu = K_\nu - K_0) \) IS POSITIVE DEFINITE (ONE CAUTION SEE (2))

THEN: FOR A GUESSED SET \( \alpha_\nu \) ONE CAN FIND \( \psi(x,y) \)

(USING JACOBI ALGORITHM); SUBSEQUENTLY ONE CAN FIND THE \( K_\nu \)'s (AND THE \( \delta K_\nu \)'s); THEN ONE CAN FIND A BETTER SET OF \( \alpha_\nu \)'s WITHOUT INVERTING \( M_{\psi,\psi} \).
NUMERICS SPECIFICS:

A SET OF FIXED $\psi$-VALUES ARE CHOSEN:

\[ \psi_{\text{num}} \rightarrow \psi_{\text{nu}} \]

$\psi_{\text{num}+1} > \psi_{\text{nu}}$; $\psi_1 = 0.0$; $\psi_{\text{num}+1} \approx 1.0$

ALMOST SAME NUMBER OF GRID POINTS WITH

$\psi_{\text{nu}} < \psi_{\text{nu}} \ast < \psi_{\text{num}+1}$ \quad (NU < NMAX - 2)

THE SET $g_\psi(\psi)$ IS CHOSEN SO THAT ($G_U = G_{\text{nu}}$)

\[
\frac{dG_{\text{nu}}}{d\psi} = \begin{cases} 
\frac{\psi - \psi_{\text{nu}}}{\psi_{\text{nu}} - \psi_{\text{nu}-1}} & \text{for } \psi_{\text{nu}-1} < \psi < \psi_{\text{nu}} \\
1 - \frac{\psi - \psi_{\text{nu}}}{\psi_{\text{nu}+1} - \psi_{\text{nu}}} & \text{for } \psi_{\text{nu}} < \psi < \psi_{\text{nu}+1} \\
0 & \text{otherwise}
\end{cases}
\]

INITIAL (UNFORCED) EQUILIBRIUM: $\psi = \frac{\cot k - \cot k_0}{1 - \cot k_0}$; $k_0 = k^2 - (\beta k)^2$

$\Delta' = -2k \frac{\cot k_0}{\sin k_0}$. INITIAL EQUILIBRIA CONSIDERED DEPEND ON $\Delta'$ AND ASPECT RATIO, $b/a$ ONLY.
THREE STEPS ARE TAKEN AT EACH ITERATION:

1. JACOBI STEP:
   \[ \psi_{n+1} = \psi_N + c_1 \left[ \psi_N + \sum_u \alpha_u \frac{\partial \psi}{\partial u} \right] \]

2. IMPROVE \( \alpha_u \)S:
   \[ \alpha_u^{n+1} = \alpha_u^N + c_2 \left[ k_u^N - k \right] \]

3. IMPOSE EXTERNAL CURRENTS (BOUNDARY CONDITION)
   \[ \psi_{n+1}^{\text{after}}(x, b) = \frac{\psi_{n+1}^{\text{before}}(x, b)}{\max \left\{ \psi_{\text{before}}(x, b) \right\}} \cdot \psi_{\text{sep}} \]

THE EXTERNAL (FORCING) CURRENT (RELATIVE TO TOTAL PLASMA CURRENT) IS APPROXIMATELY (SIGN IS CORRECT)

\[ J_x = \left[ \frac{\max \left\{ \psi_{\text{before}}(x, 0) \right\}}{\max \left\{ \psi_{\text{before}}(x, b) \right\}} \cdot \psi_{\text{sep}} - 1 \right] a \]

\( c_1 = 0.2 \Delta^2 \) (DIS GRID DISTANCE); \( c_2 \) IS ADJUSTED FOR GOOD CONVERGENCE. GOOD (EXPONENTIAL) CONVERGENCE WAS FOUND IN PARAMETER SPACE EXPLORED.

OBJECT OF CALCULATION IS TO FIND:

\[ J_x(\Delta a, b/a, \psi_{\text{sep}}, \text{NUMAX}) \]
EXAMPLE OF CONVERGENCE OF ALGORITHM:

\[ \text{Iteration Number} \]

\[ \begin{align*}
\text{max} &= 61 \\
\text{mean} &= 118 \\
\text{std} &= 30 \\
\text{num} &= 10
\end{align*} \]
SENSITIVITY TO GRID SIZE

\[
\frac{b}{a} = 3.3 \\
\begin{cases}
    \Delta = 2.0; Y_\text{gap} = 0.985; \nu_\text{max} = 20 \\
    \Delta = 2.0; Y_\text{gap} = 0.38; \nu_\text{max} = 10
\end{cases}
\]

CHOICE: \( \nu_\text{max} = 61 \); \( \nu_\text{max} = 118 \)
If, for a real plasma, the forcing current were suddenly removed, a singular current with the same sign as the forcing current will appear at the separatrix. With any resistivity present, the $\gamma$-value at the separatrix will increase or decrease depending on sign of the singular current.

Signs are chosen so that a negative $J_x$ makes $\gamma$ at separatrix increase $\rightarrow$ island will shrink.

Conversely, a positive $J_x$ means a growing island.

For $J_x = 0$ the equilibrium is stable if $\frac{\partial J_x}{\partial \Phi_p} > 0$, a saturated island equilibrium. For $\frac{\partial J_x}{\partial \Phi_p} < 0$ equilibrium is unstable.
RESULTS:

i) $\Delta a = 0.0$ is stability limit for initial equilibrium.

ii) For $\Delta a > 0.0$ the saturated island width increases with $\Delta a$.

$N_{\text{MAX}} = 10$, $b/a = 3.8$

\[ \Delta a \begin{cases} 4.0 \\ -2.0 \\ 0.0 \\ 2.0 \end{cases} \]

[iii] $J_x$ is insensitive to aspect ratio $b/a$.

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<tr>
<th>$b/a$</th>
<th>$\psi_{\text{sep}}$</th>
<th>2.5</th>
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<td>0.995</td>
<td>$3.24 \times 10^{-3}$</td>
<td>$3.01 \times 10^{-3}$</td>
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The saturated island width is sensitive to NUMAX, i.e., the assumptions on how far circumstancies are from those of ideal MHD. (Surprise 2, 3, 4, 5)

\[ \Delta' \alpha = 2.0, \quad b/\lambda = 3.8 \]

Ref. 3 White, Monticello, Rosenbluth + Wadell, Phys. of Fluids, 20, 800 (1977)
Ref. 4 Carreras, Wadell + Hicks, Nucl. Fus. 19, 1423 (1979)
Ref. 5 Ren, Jensen + Calle, Phys. of Plasmas 5, 2574 (1998)

Acknowledgment: Joe Freeman made this work possible by his kind and infinitely patient way of teaching me some FORTRAN.
RESULTS:

i) \( \Delta a = 0.0 \) is STABILITY LIMIT FOR INITIAL EQUILIBRIUM

ii) For \( \Delta a > 0.0 \) THE SATURATED ISLAND WIDTH INCREASES WITH \( \Delta a \)

\[
\text{NUMAX} = 10, \quad \frac{b}{a} = 2.0
\]

iii) \( J_x \) IS INSENSITIVE TO ASPECT RATIO \( (b/a) \); \( \Delta a = 2.0 \)

\[
\text{NUMAX} \geq 10
\]

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