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Kinetic Theory of Tokamak Zonal Flow Dynamics

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- We consider the nonlinear interaction of two groups of fluctuating potentials in tokamaks.
  - Axisymmetric potentials, which include zonal flows
  - Nonaxisymmetric potentials, which include drift waves
- A gyrokinetic description is essential for the axisymmetric potentials.
  - We have shown that, for times longer than an ion bounce time, residual flows develop, which are linearly damped only by collisions.
  - What is the nonlinear damping of the zonal flows ?
- A gyrofluid description may be adequate for the nonaxisymmetric potentials.
  - How to include these potentials in the axisymmetric kinetic description ?
  - What role do the zonal flows play in the drift wave dynamics ?
- We answer these questions using a simple model.

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Kinetic Theory of Tokamak Zonal Flow Dynamics[1] F.L. HINTON, M.N. ROSENBLUTH, General Atomics, P.H. DIAMOND, University of California, San Diego, L. CHEN, University of California, Irvine — The nonlinear interaction of drift wave turbulence with axisymmetric potentials (sheared $E \times B$ or "zonal" flows) is investigated. Starting with the gyrokinetic equation in toroidal geometry, the axisymmetric linear response is determined, including both geodesic acoustic modes and collisionally damped residual flows. These flows are driven nonlinearly by the drift wave turbulence. An equation for the drift wave potentials is derived by using a simple ion fluid closure and assuming adiabatic electrons. For ion gyroradius much smaller than the wavelength, this is equivalent to a polarization drift nonlinearity. An electron nonlinearity also exists because the electron response, to the axisymmetric potentials, is not adiabatic. The electron nonlinearity is essential for energy conservation in the coupled equations for the potentials. Weak turbulence theory is used to derive equations for the drift wave and zonal flow intensities.

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Nonlinear Interaction in a Cylindrical Model

In the electrostatic approximation, the potential is determined by quasineutrality:

$$n_0 \sum \frac{e^2}{T} \phi(\vec{x}, t) = \sum e \int d^3v \, G(\vec{x}, \vec{v}, t) \tag{1}$$

where

$$G(\vec{x}, \vec{v}, t) = \oint \frac{d\alpha}{2\pi} g(\vec{x} - \vec{\rho}, \vec{v}, t) \tag{2}$$

where α is the gyro-angle, $\rho = v_\perp / \Omega$, and where g is a solution of the gyrokinetic equation (GKE):

$$\frac{\partial g}{\partial t} + v_z \frac{\partial g}{\partial z} - \mathcal{C} g = \frac{e}{T} F_0 \frac{\partial \Phi}{\partial t} - \frac{c}{B} \hat{e}_z \times \nabla \Phi \cdot \nabla F_0 + S \tag{3}$$

where $\mathcal{C}$ is the collision operator and S is the $E \times B$ nonlinearity:

$$S = -\frac{c}{B} \hat{e}_z \times \nabla \Phi \cdot \nabla g \tag{4}$$

The gyro-averaged potential is defined by

$$\Phi(\vec{R}, v_\perp, t) = \oint \frac{d\alpha}{2\pi} \phi(\vec{R} + \vec{\rho}, t) \tag{5}$$

Electrons are Nonadiabatic for Zonal Flows

We write the potential as the sum of two parts:

$$\phi = \overline{\phi} + \tilde{\phi} \qquad (6)$$

where $\overline{\phi}$ is the average over θ and z, the zonal flow potential, and where $\tilde{\phi}$ is the θ and z - dependent drift wave potential.
The electron GKE, neglecting collisions, is

$$\frac{\partial g_e}{\partial t} + v_z \frac{\partial g_e}{\partial z} = -\frac{e}{T_e} F_{e0} \frac{\partial \phi}{\partial t} - \frac{c}{B} \hat{e}_z \times \nabla \phi \cdot \nabla (F_{e0} + g_e) \qquad (7)$$

Taking the average over θ and z, we have

$$\frac{\partial \overline{g_e}}{\partial t} = -\frac{e}{T_e} F_{e0} \frac{\partial \overline{\phi}}{\partial t} - \frac{c}{B} \overline{\hat{e}_z \times \nabla \tilde{\phi} \cdot \nabla \tilde{g}_e} \qquad (8)$$

where $g_e = \overline{g_e} + \tilde{g}_e$, as with the potential.

Assuming $\omega/(k_{\|}v_e) \ll 1$ we expand in this small parameter to obtain

$$\tilde{g}_e \simeq 0 \tag{9}$$

and therefore $\tilde{n}_e = n_0(e/T_e)\tilde{\phi}$ (adiabatic electrons).
Using this in Eq.(8), we find

$$\overline{g_e} \simeq -\frac{e}{T_e}F_{e0}\overline{\phi} \tag{10}$$

and therefore $\overline{n_e} = n_0$ (nonadiabatic electrons).

Quasineutrality thus has two different forms for $\tilde{\phi}$ and $\overline{\phi}$:

$$\frac{n_0 e}{T_i}(1+\frac{T_i}{T_e})\tilde{\phi} = \int d^3v\ \tilde{G}_i \tag{11}$$

and

$$\frac{n_0 e}{T_i}\overline{\phi} = \int d^3v\ \overline{G_i} \tag{12}$$

Time Derivative of Quasineutrality

Taking the time derivative,

$$\frac{n_0 e}{T_i}\frac{\partial}{\partial t}\bar{\phi} = \int d^3v \, \frac{\partial}{\partial t} \bar{G}_i \tag{13}$$

and

$$\frac{n_0 e}{T_i}\left(1+\frac{T_i}{T_e}\right)\frac{\partial}{\partial t}\tilde{\phi} = \int d^3v \, \frac{\partial}{\partial t}\tilde{G}_i \tag{14}$$

We now use Fourier transforms:

$$\tilde{\phi} \to \phi_{\vec{k}} \; , \; \bar{\phi} \to \phi_{\vec{q}} \tag{15}$$

where $\vec{k} = (k_r, k_\theta, k_z)$ and $\vec{q} = (q_r, 0, 0)$. Then

$$\frac{n_0 e}{T_i}\frac{\partial \phi_{\vec{q}}}{\partial t} = \int d^3v \; J_0(q_r\rho)\frac{\partial g_{i\vec{q}}}{\partial t} \tag{16}$$

and

$$\frac{n_0 e}{T_i}\left(1+\frac{T_i}{T_e}\right)\frac{\partial \phi_{\vec{k}}}{\partial t} = \int d^3v \; J_0(k_\perp\rho)\frac{\partial g_{i\vec{k}}}{\partial t} \tag{17}$$

Zonal Flow Equation

The Fourier transform of the average of the GKE is

$$\frac{\partial g_{i\vec{q}}}{\partial t} - \mathcal{C}_{ii} g_{i\vec{q}} = \frac{e}{T_i} F_{i0} J_0(q_r \rho) \frac{\partial \phi_{\vec{q}}}{\partial t} + S_{i\vec{q}} \tag{18}$$

Using this in the time derivative of the symmetric quasineutrality equation, we obtain the equation for the zonal flow potentials:

$$\chi_{\vec{q}} \left(\frac{\partial \phi_{\vec{q}}}{\partial t} + \nu_q \phi_{\vec{q}} \right) = \frac{T_e}{n_0 e} \int d^3v \; J_0(q_\perp \rho) S_{i\vec{q}} \tag{19}$$

where ν_q is the collisional viscous damping rate,

$$\nu_q = -\frac{\int d^3v \; J_0(q_r \rho) \mathcal{C}_{ii} \left(J_0(q_r \rho) F_{i0} \right)}{n_0 \Gamma_0(b_q) \left(1 - \Gamma_0(b_q)\right)} \tag{20}$$

and where the linear susceptibility is

$$\chi_{\vec{q}} = \frac{T_e}{T_i} \left(1 - \Gamma_0(b_q)\right) \tag{21}$$

with $\Gamma_0(b) = e^{-b} I_0(b)$, where I_0 is a modified Bessel function, $b_q = q_r^2 a_i^2$, and $a_i = (T_i/m_i)^{1/2}/\Omega_i$.

Drift Wave Equation

Assuming $k_{||} v_i \ll \omega$ and neglecting collisions, the $\vec{k}$ component of the GKE is

$$\frac{\partial g_{i\vec{k}}}{\partial t} = \frac{e}{T_i} F_{i0} J_0(k_\perp \rho) \left[\frac{\partial \phi_{\vec{k}}}{\partial t} - \frac{T_i}{T_e} \vec{k} \cdot \vec{v}_* \phi_{\vec{k}} \right] + S_{i\vec{k}} \tag{22}$$

where $\vec{v}_* = -(cT_e/eB)\hat{e}_z \times \nabla \ln n_0$. Using this in the time derivative of the quasineutrality equation, we obtain the equation for the drift wave potentials:

$$\chi_{\vec{k}} \left(\frac{\partial \phi_{\vec{k}}}{\partial t} + i\omega_{\vec{k}} \phi_{\vec{k}} \right) = \frac{T_e}{n_0 e} \int d^3v \; J_0(k_\perp \rho) S_{i\vec{k}} \tag{23}$$

where the linear susceptibility is

$$\chi_{\vec{k}} = 1 + \frac{T_e}{T_i} \left(1 - \Gamma_0(b_k)\right) \tag{24}$$

with $b_k = k_\perp^2 a_i^2$, and the linear frequency is

$$\omega_{\vec{k}} = \frac{\Gamma_0(b_k)}{\chi_{\vec{k}}} \vec{k} \cdot \vec{v}_* \tag{25}$$

E x B Nonlinearities

The nonlinear terms are the Fourier transforms of the ExB nonlinearities in the gyrokinetic equation:

$$S_{i\vec{q}} = (c/2B) \sum_{\vec{k}'(\vec{k}'' \equiv \vec{q} - \vec{k}')} \hat{e}_z \cdot \vec{k}' \times \vec{k}'' \left[J_0(k'_\perp \rho) \phi_{\vec{k}'} g_{i\vec{k}''} - J_0(k''_\perp \rho) \phi_{\vec{k}''} g_{i\vec{k}'} \right] \tag{26}$$

Also

$$\begin{aligned} S_{i\vec{k}} = {} & (c/2B) \sum_{\vec{k}'(\vec{k}'' \equiv \vec{k} - \vec{k}')} \hat{e}_z \cdot \vec{k}' \times \vec{k}'' \left[J_0(k'_\perp \rho) \phi_{\vec{k}'} g_{i\vec{k}''} - J_0(k''_\perp \rho) \phi_{\vec{k}''} g_{i\vec{k}'} \right] \\ & + (c/B) \sum_{\vec{q}'(\vec{k}'' \equiv \vec{k} - \vec{q}')} \hat{e}_z \cdot \vec{q}' \times \vec{k}'' \left[J_0(q'_r \rho) \phi_{\vec{q}'} g_{i\vec{k}''} - J_0(k''_\perp \rho) \phi_{\vec{k}''} g_{i\vec{q}'} \right] \end{aligned} \tag{27}$$

Approximate Fluid Closure Relations

To evaluate the nonlinear terms, we need approximate relations between $g_{i\vec{k}}$, $g_{i\vec{q}}$ and $\phi_{i\vec{k}}$, $\phi_{i\vec{q}}$. We use a simple fluid approximation for the ions:

$$F_i = F_{i0}\left[1 + \delta n_i(\vec{x}, t)/n_0\right] \tag{28}$$

Then, using $g(\vec{R}, \vec{v}, t) = F_i(\vec{R} + \vec{\rho}, \vec{v}, t) + (e/T_i)F_{i0}\phi(\vec{R} + \vec{\rho}, \vec{v}, t)$, we have

$$g_{i\vec{k}} = J_0[F_{i\vec{k}} + (e/T_i)F_{i0}\phi_{\vec{k}}] = J_0F_{i0}\left[\delta n_{i\vec{k}}/n_0 + (e/T_i)\phi_{\vec{k}}\right] \tag{29}$$

Using quasineutrality with adiabatic electrons,

$$\begin{aligned} \frac{n_0 e}{T_i}\left(1 + \frac{T_i}{T_e}\right)\phi_{\vec{k}} &= \int d^3v\; J_0(k_\perp\rho)g_{i\vec{k}} \\ &= n_0\Gamma_0(b_k)\left[\frac{\delta n_{i\vec{k}}}{n_0} + \frac{e}{T_i}\phi_{\vec{k}}\right] \end{aligned} \tag{30}$$

we find

$$g_{i\vec{k}} = \frac{e}{T_i}\left(1 + \frac{T_i}{T_e}\right)F_{i0}\frac{J_0(k_\perp\rho)}{\Gamma_0(b_k)}\phi_{\vec{k}} \tag{31}$$

Similarly for the symmetric potentials: using

$$g_{i\vec{q}} = J_0 F_{i0} \left[\delta n_{i\vec{q}}/n_0 + (e/T_i)\phi_{\vec{q}}\right] \tag{32}$$

and the appropriate quasineutrality equation,

$$\frac{n_0 e}{T_i}\phi_{\vec{q}} = \int d^3v \; J_0(q_r\rho) g_{i\vec{q}} \tag{33}$$

to determine $\delta n_{i\vec{q}}$, we obtain

$$g_{i\vec{q}} = \frac{e}{T_i} F_{i0} \frac{J_0(q_r\rho)}{\Gamma_0(b_q)} \phi_{\vec{q}} \tag{34}$$

Mode Coupling Equations

Substituting the closure relation for $g_{i\vec{k}}$ into Eqs.(19) and (26), we find

$$\chi_{\vec{q}}\left(\frac{\partial \phi_{\vec{q}}}{\partial t}+\nu_q \phi_{\vec{q}}\right)=\frac{1}{2}\left(1+\frac{T_i}{T_e}\right)\sum_{\vec{k}'} M(\vec{k}', \vec{q}-\vec{k}')\phi_{\vec{k}'}\phi_{\vec{q}-\vec{k}'} \quad (35)$$

where

$$M(\vec{k}_1, \vec{k}_2)=\frac{cT_e}{BT_i}(\hat{e}_z \cdot \vec{k}_1 \times \vec{k}_2)\mathcal{I}(\vec{k}_1, \vec{k}_2)\left[\frac{1}{\Gamma_0(b_2)}-\frac{1}{\Gamma_0(b_1)}\right] \quad (36)$$

with $b_1 = k_{1\perp}^2 a_i^2$ and $b_2 = k_{2\perp}^2 a_i^2$, and

$$\mathcal{I}(\vec{k}_1, \vec{k}_2)=\frac{1}{n_0}\int d^3v\ F_{i0} J_0(k_{3\perp}\rho) J_0(k_{1\perp}\rho) J_0(k_{2\perp}\rho) \quad (37)$$

in which k_3 is defined by $\vec{k}_1+\vec{k}_2+\vec{k}_3=0$.

Similarly, substituting the closure relations for $g_{i\vec{k}}$ and $g_{i\vec{q}}$ into Eqs.(23) and (27), we find

$$\chi_{\vec{k}}\left(\frac{\partial\phi_{\vec{k}}}{\partial t}+i\omega_{\vec{k}}\phi_{\vec{k}}\right)=\frac{1}{2}\left(1+\frac{T_i}{T_e}\right)\sum_{\vec{k}'}M(\vec{k}',\vec{k}-\vec{k}')\phi_{\vec{k}'}\phi_{\vec{k}-\vec{k}'} + \sum_{\vec{q}'}N(\vec{q}',\vec{k}-\vec{q}')\phi_{\vec{q}'}\phi_{\vec{k}-\vec{q}'} \tag{38}$$

where

$$N(\vec{q},\vec{k})=M(\vec{q},\vec{k})+\frac{c}{B}(\hat{e}_z\cdot\vec{q}\times\vec{k})\frac{\mathcal{I}(\vec{q},\vec{k})}{\Gamma_0(b_k)} \tag{39}$$

Eqs.(35) - (39) generalize to arbitrary ion gyroradius the equations for zonal flow and drift wave potentials given by Smolyakov, Diamond and Malkov, 1999 (to be published in P.R.L.)

The Electron Nonlinearity

In the long wavelength limit, $k_\perp^2 a_i^2 \ll 1$ and $q_r^2 a_i^2 \ll 1$, the matrix element for drift wave - zonal flow scattering is approximately

$$N(\vec{q}, \vec{k}) \simeq \frac{c}{B} \hat{e}_z \cdot \vec{q} \times \vec{k} \qquad \cancel{(49)}$$

This can be identified with the term

$$\frac{c}{B} \hat{e}_z \cdot \nabla \phi \times \nabla \tilde{\phi} \qquad \cancel{(50)}$$

which must be added to the Hasagawa-Mima equation to account for the nonadiabatic electron response to the zonal flows: see Smolyakov, et al, 1999. Since this nonlinearity is not small in $k_\perp^2 a_i^2$ at long wavelengths,
it dominates the drift wave - zonal flow scattering.

Energy Conservation

Multiplying Eq.(35) by $\phi^*_{\vec{q}}$ and summing over $\vec{q}$, we obtain

$$\frac{\partial}{\partial t}\sum_{\vec{q}}\chi_{\vec{q}}\phi^*_{\vec{q}}\phi_{\vec{q}} = -2\sum_{\vec{q}}\nu_q\chi_{\vec{q}}\phi^*_{\vec{q}}\phi_{\vec{q}} + \frac{1}{2}\left(1+\frac{T_i}{T_e}\right)\sum_{\vec{q},\vec{k}} M(\vec{k},\vec{q}-\vec{k})\phi^*_{\vec{q}}\phi_{\vec{k}}\phi_{\vec{q}-\vec{k}} + c.c. \quad (40)$$

(where $c.c.$ means complex conjugate) Similarly, multiplying Eq.(38) by $\phi^*_{\vec{k}}$ and summing over $\vec{k}$, we obtain

$$\frac{\partial}{\partial t}\sum_{\vec{k}}\chi_{\vec{k}}\phi^*_{\vec{k}}\phi_{\vec{k}} = 2\sum_{\vec{k}}\gamma_{\vec{k}}\chi_{\vec{k}}\phi^*_{\vec{k}}\phi_{\vec{k}} + \frac{1}{2}\left(1+\frac{T_i}{T_e}\right)\sum_{\vec{k},\vec{k}'} M(\vec{k}',\vec{k}-\vec{k}')\phi^*_{\vec{k}}\phi_{\vec{k}'}\phi_{\vec{k}-\vec{k}'} + c.c. + \sum_{\vec{k},\vec{q}} N(\vec{q},\vec{k}-\vec{q})\phi^*_{\vec{k}}\phi_{\vec{q}}\phi_{\vec{k}-\vec{q}} + c.c. \quad (41)$$

where γ_k, the linear growth rate, is now included *ad hoc*.

We use the symmetry property of M,

$$M(\vec{k}_1, \vec{k}_2) + M(\vec{k}_2, \vec{k}_3) + M(\vec{k}_3, \vec{k}_1) = 0 \tag{42}$$

where $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$, and the reality condition $\phi_{-\vec{k}} = \phi^*_{\vec{k}}$ to show that the drift wave - drift wave scattering conserves energy:

$$Re \sum_{\vec{k},\vec{k}'} M(\vec{k}', \vec{k} - \vec{k}')\phi^*_{\vec{k}}\phi_{\vec{k}'}\phi_{\vec{k}-\vec{k}'} = 0 \tag{43}$$

Also, using the relation between N and M,

$$N(\vec{k}_1, \vec{k}_2) + N(\vec{k}_1, \vec{k}_3) + \left(1 + \frac{T_i}{T_e}\right) M(\vec{k}_2, \vec{k}_3) = 0 \tag{44}$$

we find that the drift wave - zonal flow scattering also conserves energy:

$$\left(1 + \frac{T_i}{T_e}\right) Re \sum_{\vec{q},\vec{k}} M(\vec{k}, \vec{q} - \vec{k})\phi^*_{\vec{q}}\phi_{\vec{k}}\phi_{\vec{q}-\vec{k}} + 2Re \sum_{\vec{k},\vec{q}} N(\vec{q}, \vec{k} - \vec{q})\phi^*_{\vec{k}}\phi_{\vec{q}}\phi_{\vec{k}-\vec{q}} = 0 \tag{45}$$

Therefore, the energy is changed only by the linear damping and growth:

$$\frac{\partial}{\partial t}\left[\sum_{\vec{q}} \chi_{\vec{q}} \phi^*_{\vec{q}} \phi_{\vec{q}} + \sum_{\vec{k}} \chi_{\vec{k}} \phi^*_{\vec{k}} \phi_{\vec{k}}\right] = -\sum_{\vec{q}} \nu_q \chi_{\vec{q}} \phi^*_{\vec{q}} \phi_{\vec{q}} + \sum_{\vec{k}} \gamma_{\vec{k}} \chi_{\vec{k}} \phi^*_{\vec{k}} \phi_{\vec{k}} \tag{46}$$

A steady state, achieved by balancing the linear growth of the drift waves with the collisional damping of the zonal flows, implies

$$\frac{|\phi_{\vec{k}}|^2}{|\phi_{\vec{q}}|^2} \sim \frac{\nu_q \chi_{\vec{q}}}{\gamma_{\vec{k}} \chi_{\vec{k}}} \ll 1 \tag{47}$$

We have assumed $\gamma_{\vec{k}} \sim \omega_{\vec{k}} \sim k_\perp \rho_s c_s / L_n$, and also that the collisional damping ν_q is small:

$$\frac{\nu_q}{\Omega_i} \ll \frac{1}{k_\perp L_n} \sim \frac{1}{q_r L_n} \ll 1 \tag{48}$$

Therefore, the zonal flow potentials are much larger than the drift wave potentials.

The Dominant Mode Coupling Terms

The equations for the dimensionless Fourier amplitudes ($\phi \equiv e\phi/T_e$) are

$$\frac{\partial \phi_{\vec{q}}}{\partial t} + \nu_q \phi_{\vec{q}} = \frac{1}{2} \sum_{\vec{k}'} \Lambda^{ZD}_{\vec{k}',\vec{q}-\vec{k}'} \phi_{\vec{k}'} \phi_{\vec{q}-\vec{k}'} \tag{51}$$

$$\frac{\partial \phi_{\vec{k}}}{\partial t} + (i\omega_{\vec{k}} - \gamma_{\vec{k}})\phi_{\vec{k}} = \frac{1}{2} \sum_{\vec{q}'} \Lambda^{DZ}_{\vec{q}',\vec{k}-\vec{q}'} \phi_{\vec{q}'} \phi_{\vec{k}-\vec{q}'} \tag{52}$$

where we have defined

$$\Lambda^{ZD}_{\vec{k}',\vec{k}''} = \frac{T_e}{e}\left(1 + \frac{T_i}{T_e}\right) \frac{M(\vec{k}', \vec{k}'')}{\chi_{\vec{k}'+\vec{k}''}} \tag{53}$$

and

$$\Lambda^{DZ}_{\vec{q},\vec{k}} = \frac{2T_e}{e} \frac{N(\vec{q}, \vec{k})}{\chi_{\vec{q}+\vec{k}}} \tag{54}$$

The mode coupling terms in Eq.(38) involving $\tilde{\phi}_{\vec{k}'} \tilde{\phi}_{\vec{k}-\vec{k}'}$ have been neglected because they are smaller than the terms involving $\tilde{\phi}_{\vec{q}'} \tilde{\phi}_{\vec{k}-\vec{q}'}$ by a factor of order $k_\perp^2 \rho_s^2 |\phi_{\vec{k}}/\phi_{\vec{q}}| \ll 1$, for small $k_\perp \rho$.

Zonal Flow Intensity Equation

We use the weak coupling approximation (Kadomtsev, Plasma Turbulence, p.49) to obtain equations for the intensities $I_{\vec{q}} \equiv \left\langle |\phi_{\vec{q}}|^2 \right\rangle$ and $I_{\vec{k}} \equiv \left\langle |\phi_{\vec{k}}|^2 \right\rangle$:

$$\frac{\partial I_{\vec{q}}}{\partial t} + (2\nu_q + \eta_{\vec{q}}) I_{\vec{q}} = \sum_{\vec{k}'} \lambda_{\vec{q},\vec{k}'} I_{\vec{k}'} I_{\vec{q}-\vec{k}'} \tag{55}$$

where the nonlinear damping rate is

$$\eta_{\vec{q}} = -\sum_{\vec{k}'} \frac{\Lambda^{ZD}_{\vec{k}',\vec{q}-\vec{k}'} \Lambda^{DZ}_{\vec{q},\vec{k}'-\vec{q}} \left(\Gamma_{\vec{k}'-\vec{q}} + \Gamma_{\vec{q}}\right)}{\omega_{\vec{k}'-\vec{q}}{}^2 + \left(\Gamma_{\vec{k}'-\vec{q}} + \Gamma_{\vec{q}}\right)^2} I_{\vec{k}'-\vec{q}} \tag{56}$$

and the nonlinear excitation coefficient is

$$\lambda_{\vec{q},\vec{k}'} = \frac{\left(\Lambda^{ZD}_{\vec{k}',\vec{q}-\vec{k}'}\right)^2 \left(\Gamma_{\vec{k}'} + \Gamma_{\vec{q}-\vec{k}'}\right)}{\left(\omega_{\vec{k}'} - \omega_{\vec{k}'-\vec{q}}\right)^2 + \left(\Gamma_{\vec{k}'} + \Gamma_{\vec{q}-\vec{k}'}\right)^2} \tag{57}$$

with $\Gamma_{\vec{k}}$ and $\Gamma_{\vec{q}}$ the assumed decorrelation rates for the potentials $\phi_{\vec{k}}$ and $\phi_{\vec{q}}$.

Nonlinear "Damping" of Zonal Flows

In the long wavelength limit,

$$\Lambda^{ZD}_{\vec{k}',\vec{q}-\vec{k}'}\Lambda^{DZ}_{\vec{q},\vec{k}'-\vec{q}} \simeq 2\rho_s^2 c_s^2 \left(1+\frac{T_i}{T_e}\right)\frac{(\hat{e}_z\cdot\vec{k}'\times\vec{q})^2(q_r^2-2k_r'q_r)}{q_r^2} \tag{58}$$

the nonlinear damping rate of zonal flows has a simple dependence on q_r:

$$\eta_{\vec{q}} \propto q_r^2 \tag{59}$$

Also, this damping rate is negative:

$$\eta_{\vec{q}} < 0 \tag{60}$$

so the zonal flows are nonlinearly unstable. In fact, since the λ terms in Eq.(55) are small, we must have

$$\eta_{\vec{q}} \simeq -2\nu_q \left(\propto q_r^2 \text{, since } \nu_q \simeq \nu_{ii}\rho_i^2 q_r^2\right) \tag{61}$$

Therefore, the saturated drift wave intensity is proportional to the collision frequency:

$$I_{\vec{k}} \propto \nu_{ii} \tag{62}$$

which agrees with the gyrokinetic particle simulations of Z. Lin, et al, (P.R.L. Nov. 1, 1999), and Diamond, et al. IAEA 1998.

Drift Wave Intensity Equation

$$\frac{\partial I_{\vec{k}}}{\partial t} + \eta_{\vec{k}} I_{\vec{k}} = 2\gamma_{\vec{k}} I_{\vec{k}} + \sum_{\vec{q}'} \lambda_{\vec{k},\vec{q}'} I_{\vec{q}'} I_{\vec{k}-\vec{q}'} \tag{63}$$

where the nonlinear damping rate is

$$\eta_{\vec{k}} = -\frac{1}{2} \sum_{\vec{q}'} \frac{\Lambda^{DZ}_{\vec{q}',\vec{k}-\vec{q}'} \Lambda^{DZ}_{-\vec{q}',\vec{k}} \left(\Gamma_{\vec{k}} + \Gamma_{\vec{q}'}\right)}{\omega_{\vec{k}}{}^2 + \left(\Gamma_{\vec{k}} + \Gamma_{\vec{q}'}\right)^2} I_{\vec{q}'} \tag{64}$$

(we have neglected smaller contributions proportional to $I_{\vec{k}-\vec{q}'}$) and the nonlinear excitation coefficient is

$$\lambda_{\vec{k},\vec{q}'} = \frac{1}{2} \frac{\left(\Lambda^{DZ}_{\vec{q}',\vec{k}-\vec{q}'}\right)^2 \left(\Gamma_{\vec{q}'} + \Gamma_{\vec{k}-\vec{q}'}\right)}{\omega_{\vec{k}-\vec{q}'}{}^2 + \left(\Gamma_{\vec{q}'} + \Gamma_{\vec{k}-\vec{q}'}\right)^2} \tag{65}$$

Nonlinear Damping of Drift Waves

Using the dominant approximation $N(\vec{q},\vec{k}) \simeq (c/B)\hat{e}_z \cdot \vec{q} \times \vec{k}$, we have

$$\Lambda^{DZ}_{\vec{q}',\vec{k}-\vec{q}'}\Lambda^{DZ}_{-\vec{q}',\vec{k}} \simeq -4\rho_s^2 c_s^2(\hat{e}_z \cdot \vec{q}' \times \vec{k})^2 < 0 \qquad (66)$$

Therefore, the nonlinear drift wave damping is positive:

$$\eta_{\vec{k}} > 0 \qquad (67)$$

i.e., drift waves are damped by scattering from zonal flows. This is the mechanism of shear suppression of turbulence, within this simple model.

Conclusions

- For long wavelengths and small collision frequency, the zonal flow potentials are much larger than the drift wave potentials.
- The nonlinear "damping" of zonal flows is given by Eq.(56); for long wavelengths and $k_r \ll q_r$, we find $\eta_{\vec{q}} \propto q_r^2$ and $\eta_{\vec{q}} < 0$ (nonlinear instability).
- The nonlinear damping of drift waves is given by Eq.(64); for long wavelengths and $k_r \ll q_r$, we find $\eta_{\vec{k}} > 0$ (shear suppression).
- The saturated drift wave intensity is proportional to the ion-ion collision frequency: $I_{\vec{k}} \propto \nu_{ii}$.