EXTENSION OF GATO — AN IDEAL MHD STABILITY CODE FOR AXISYMMETRIC PLASMA EQUILIBRIA TO BALLOONING MODE VARIABLES VALID FOR HIGHER (n > 5) TOROIDAL MODE NUMBERS

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Stability of Finite-n Global Magnetohydrodynamic Modes Using the GATO Stability Code<sup>1</sup> M.S. CHU, S.K. WONG, L.L. LAO, A.D. TURNBULL, General Atomics, M.S. CHANCE, Princeton Plasma Physics Laboratory — This work extends the capability of the GATO stability code<sup>2</sup> to analyze realistic numerical tokamak equilibria for their stability to higher  $n ~(\sim 5-10)$  MHD modes. This is motivated by the experimental evidence of these modes being relevant for both plasma termination and the behavior of ELMs. The ballooning angle transformation<sup>3</sup> is applied to the displacement variables in the GATO representation. The potential energy matrix is constructed with the inclusion of extra mapping quantities. The vacuum energy computed from the Greens function is also modified to couple to the transformed displacement at the plasma boundary. The resultant eigenvalue problem is solved with the modified boundary condition in the poloidal direction suitable for these transformed variables. The dependence of the plasma stability as a function of toroidal mode number and plasma equilibrium properties will be presented.

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<sup>2</sup>L.C.Bernard *et al.*, Comput. Phys. Commun. **24**, 377 (1981).
<sup>3</sup>R. Gruber *et al.*, Comput. Phys. Commun. **24**, 363 (1981).



Prefer Oral Session Prefer Poster Session M.S. Chu chum@fusion.gat.com General Atomics

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- 1. Motivation
- 2. Formulation
- 3. Implementation
- 4. Results
  - a. Circular tokamak
  - b. D-shaped cross-section
- 5. Summary and Results

- Higher (n ~ 5) toroidal mode number MHD modes have been reported to be relevant for a number of experimental fluctuations, especially in NCS discharge at the plasma edge and also for low aspect ratio ST
- Success of correlation of these observations with either ballooning or peeling mode criterion demands a more realistic evaluation of stability of plasma to higher (n > 5)
- More intrinsic interest in the understanding of plasma behavior for higher n and non-circular cross-section especially the edge region of tokamaks with high triangularity and squareness
- The number of matrix elements required scales as  $n_{\psi}n_{\chi}^2$ . Computer time scales as  $n_{\psi}n_{\chi}^2 \ln n_{\chi}$ . For a tokamak configuration,  $n_{\psi} \sim nq$  and  $n_{\chi} \sim nq$ . Therefore the computer time scales as  $n^3q^3 \ln(nq)$ . An alternative formulation using the ballooning variables\* allows  $n_{\chi} \approx cons tant$ . Therefore the resultant computer time scales as nq.

<sup>\*</sup>R. Gruber et al., Computer Physics Communications 24, p. 363–376 (1981)

$$W_{p} = \frac{\pi}{2} \mu_{0} \int d\psi d\chi \delta W_{\ell}$$

where the local potential energy density  $\,\delta W_I$  is a sum of positive contributions and one negative term

$$\begin{split} \delta W_{\ell} &= (\delta B_{\psi})^{2} \\ &+ \left( \delta B_{\chi} + \xi_{\psi} J_{\phi} \right)^{2} \\ &+ \left( \delta B_{\phi} + \xi_{\psi} J_{\chi} \right)^{2} \\ &+ \text{ Compressional Energy} \end{split}$$

– 2JX<sup>2</sup>K Instability Drive

## $(\vec{B} \cdot \vec{\nabla})$ TERMS IN POTENTIAL ENERGY FUNCTIONAL\* FORCE THE DISPLACEMENTS (X,U,Y) TO VARY FAST IN THE POLOIDAL DIRECTION AS n INCREASES

$$\begin{split} W_{p} &= \frac{\pi}{2\mu_{0}} \int d\psi \, d\chi \, \delta W_{1} \\ \delta W_{\ell} &= \frac{1}{JB_{p}^{2}r^{2}} \left| \frac{\partial X}{\partial \chi} + \frac{inJf}{r^{2}} X \right|^{2} \\ &+ JB_{p}^{2} \left| inU - \frac{\partial X}{\partial \psi} + \frac{\mu_{0}j_{\Phi}X}{rB_{p}^{2}} - \alpha \left( \frac{\partial X}{\partial \chi} + in\frac{Jf}{r^{2}} X \right) \right|^{2} \\ &+ \frac{r^{2}}{J} \left| f \frac{\partial}{\partial \psi} \left( \frac{J}{r^{2}} \right) X + \frac{Jf}{r^{2}} \frac{\partial X}{\partial \psi} + \frac{\partial U}{\partial \chi} \right|^{2} \\ &+ \frac{\Gamma \mu_{0}p}{J} \left| J \frac{\partial X}{\partial \psi} + \frac{\partial J}{\partial \psi} X + \left( \frac{\partial Y}{\partial \chi} + in\frac{Jf}{r^{2}} Y \right) \right| \\ &+ \frac{\partial}{\partial \chi} \left( \frac{r^{2}}{f} \right) U + \frac{r^{2}}{f} \frac{\partial U}{\partial \chi} \right|^{2} \\ &- 2JX^{2} \left[ \frac{\mu_{0}^{2}j_{\Phi}^{2}}{r^{2}B_{\Phi}^{2}} - \frac{\mu_{0}j_{\Phi}}{r} \frac{\partial}{\partial \psi} \left( lnrB_{p} \right)_{v} - \mu_{0} \frac{\partial p}{\partial \psi} \frac{\partial}{\partial \psi} (lnr)_{v} \right] \end{split}$$

where

$$\mathbf{X} = \vec{\xi} \cdot \vec{\nabla} \psi \ , \quad \mathbf{Y} = \frac{\mathbf{r}}{\mathbf{f}} \boldsymbol{\xi}_{\varphi} \ , \quad \mathbf{U} = \frac{\mathbf{f}}{\mathbf{r}^2} \Big( \vec{\xi} \cdot \hat{\mathbf{t}} - \mathbf{Y} + \mathbf{J} \alpha \mathbf{X} \Big)$$

Bernard, Helton, Moore, CPC 24, p. 377 (1981).

### IN BALLOONING (HIGH n) VARIABLES, THE PLASMA DISPLACEMENT IS ASSUMED TO VARY SLOWLY ALONG THE FIELD LINE

Or the variation along the eikonal (toroidal) angle

$$\tilde{\phi} = \phi - \int_{r^2}^{\chi} \frac{fJ}{r^2} d\chi = \phi - I_q(\chi, \psi)$$

is small. The displacement is assumed to have the following dependence on this angle

$$\xi e^{in\phi} = \tilde{\xi} e^{in\tilde{\phi}}$$

Terms in the potential energy transforms in the following way

$$\frac{\partial \xi}{\partial \psi} \rightarrow \frac{\partial \tilde{\xi}}{\partial \psi} - in \tilde{\xi} I_{q,\psi}$$
$$\frac{\partial \xi}{\partial \chi} \rightarrow \frac{\partial \tilde{\xi}}{\partial \chi} - in \tilde{\xi} \frac{fJ}{r^2} \qquad \text{or}$$
$$\frac{\partial \xi}{\partial \chi} + in \frac{fJ}{r^2} \xi \rightarrow \frac{\partial \tilde{\xi}}{\partial \chi}$$

Then the potential energy functional does not contain factors of  $\left[ (\partial / \partial \chi) + in(Jf/r^2) \right]$  operating on physical quantities

## FOR GATO POTENTIAL ENERGY FUNCTIONAL IN HIGH n VARIABLES, TERMS MULTIPLIED WITH n ARE ALWAYS IN THE COMBINATION $(U + XI_{q, \Psi})$

$$\begin{split} \delta W_{\ell} &= \frac{1}{JB_{p}^{2}r^{2}} \left| \frac{\partial \tilde{X}}{\partial \chi} \right|^{2} \\ &+ JB_{p}^{2} \left| \frac{in(\tilde{U} + \tilde{X}I_{q,\psi}) - \frac{\partial \tilde{X}}{\partial \psi} + \frac{\mu_{0}j_{\phi}\tilde{X}}{rB_{p}^{2}} - \alpha \left| \frac{\partial \tilde{X}}{\partial \chi} \right|^{2} \\ &+ \frac{r^{2}}{J} \left| -in\frac{fJ}{r^{2}} (\tilde{U} + \tilde{X}I_{q,\psi}) + \frac{\partial \tilde{U}}{\partial \chi} + \frac{Jf}{r^{2}} \frac{\partial \tilde{X}}{\partial \psi} + f \left| \frac{\partial}{\partial \psi} \left( \frac{J}{r^{2}} \right) \tilde{X} \right|^{2} \\ &+ \frac{\Gamma\mu_{0}p}{J} \left| \frac{inJ(\tilde{U} + \tilde{X}I_{q,\psi})}{f} + J \frac{\partial \tilde{X}}{\partial \psi} + \frac{\partial J}{\partial \psi} \tilde{X} \right| \\ &+ \frac{\partial \tilde{Y}}{\partial \chi} + \frac{\partial}{\partial \chi} \left( \frac{r^{2}}{f} \right) \tilde{U} + \frac{r^{2}}{f} \frac{\partial \tilde{U}}{\partial \chi} \right| \\ &- 2JX^{2}K \end{split}$$

## With the ballooning angle transformation, the computation domain remains

$$\chi_0 \leq \chi \leq \chi_0 + 2\pi$$

### But the boundary condition is changed from

$$\xi(\chi_0 + 2\pi) = \xi(\chi_0)$$

#### to

$$\tilde{\xi}(\chi_0 + 2\pi) = \tilde{\xi}(\chi_0) e^{2inq(\psi)\pi}$$

The vacuum package has to be modified to accept the  $\tilde{\xi}$  as input variable

$$\delta W_{\nu} = \sum_{i,j} X_i w_{ij}^{vac} X_j$$

### transforms into

$$W_{\nu} = \sum_{i,j} \tilde{X}_{i} \tilde{w}_{ij}^{vac} \tilde{X}_{j}$$

where

$$\tilde{w}_{ij}^{vac} = e^{in[q(\chi_i)-q(\chi_j)]} w_{ij}^{vac}$$

For high n, accurate vacuum energy computation requires putting more grid points at the vacuum boundary than inner plasma surfaces

## MODIFICATIONS TO THE GATO CODE PACKAGE

- (1) Mapping: Compute  $I_q(\chi, \psi)$ ,  $I_{q,\psi}$
- (2) Matrix Elements Generation:

Compute  $\delta W_{I}$ ,  $\delta W_{v}$ 

- (3) Eigenvalue Solver: No change
- (4) Plotting: transform back from  $\tilde{\xi}$  to  $\xi$

# COMPARISON OF n = 10 MODE FOURIER HARMONICS OF ( $\xi_{\psi}, \xi_{\chi}$ )

#### With ballooning transformation only low order harmonics are large



#### Without ballooning transformation higher order harmonics present



## **COMPARISON OF GROWTH RATES**

• For the equilibrium with a circular cross-section the growth rate using ballooning transformation converges relatively fast at high n. Shown are the comparison of growth rates as a function number of flux surfaces and grid points  $(n_{\psi} \times n_{\chi})$ .



## COMPARISON OF n = 5 MODE (X)

#### With ballooning transformation



#### Without ballooning transformation



Displacement psi vs. Psi at chi (1) = 0.00417pi



## Displacement psi vs. Psi at chi (1) = 0.01250pi

# COMPARISON OF n = 10 MODE FOURIER HARMONICS OF ( $\xi_{\psi}, \xi_{\chi}$ )

## With ballooning transformation only low order harmonics are large



Without ballooning transformation higher order harmonics present



## COMPARISON OF n = 15 MODE (X)

## Rayleigh quotient = -0.1373E+00 \*\*\* Eigenvalue = -0.1373E+00 \*\*\* Error = 0.8243E-05 Maximum number of iterations = 35 \*\*\* 24 Inverse iterations done There are 0 deigenvalues less than -0.1372E+00 \*\*\* neve = 2 ntor = 15 jpsi = 300 itht = 00 rext = 1.800 qaxe = 0.80695 phi = 0.80000pi

Structure similar to low n

#### Lower harmonics

Fourier Analysis for imag Xpsi: chi = pest chi



#### **Finger like structure**

Rayleigh quotient = -0.11995-00 \*\*\*\* Eigenvalue = -0.11995+00 \*\*\*\* Error = 0.15055-04 Maximum number of iterations = 35 \*\*\*\* 35 Inverse iterations done There are 0 eigenvalues less than -0.11995+00 \*\*\*\* nevp = 1 ntor = 15 jpsi = 120 itht = 240 rext = 1.000 qaxe = 0.88695 phi = 0.00000pi



#### **Higher harmonics**



## **SUMMARY AND FUTURE DIRECTIONS**

- (1) The GATO code has been modified to the ballooning variable
- (2) Initial testing utilizing both a circular and an elongated equilibrium in DIII–D geometry indicates that this approach can be utilized to facilitate study of high (~20) n stability with modest number of grid points
- (3) The vacuum package needs to be modified to the ballooning variables for the study of peeling modes (with Chance)
- (4) Add an up-down symmetric option

## TEST CASE: A HIGH $\beta$ TOKAMAK WITH A D SHAPE CROSS–SECTION AND q<sub>0</sub> < 1



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## **COMPARISON OF GROWTH RATES**

• For the equilibrium with DIII–D cross-section the growth rate using ballooning transformation converges relatively fast at high n. Shown are the comparison of growth rates as a function of number of flux surfaces and grid points in the poloidal direction  $(n_{\psi} \times n_{\chi})$ .



## COMPARISON OF n = 5 MODE (X)

#### With ballooning transformation

Rayleigh quotient = -0.1841E-01 \*\*\* Eigenvalue = -0.1841E-01 \*\*\* Error = 0.7854E-05 Maximum number of iterations = 35 \*\*\* 11 Inverse iterations done There are 8 eigenvalues less than -0.1841E-01 \*\*\* new p = 2 ntor = 5 jpsi = 388 itht = 80 rext = 1.088 qaxe = 0.57612 phi = 0.0008pi 

Displacement psi vs. Psi at chi (1) = 0.01250pi



#### Without ballooning transformation





 Rayleigh quotient = -0.1821E-01
 \*\*\*
 Eigenvalue = -0.1821E-01
 \*\*\*
 Error = 0.1374E-05

 Maximum number of iterations = 35
 \*\*\*
 12 Inverse iterations done

 There are
 0 eigenvalues less than -0.1821E-01
 \*\*\*
 =
 1

 ntor =
 5
 jpsi = 1.20
 ith = 2.40
 rest =
 1.200
 qave =
 0.57612
 ph =
 0.0000pi



## TEST CASE: A HIGH $\beta$ TOKAMAK WITH A CIRCULAR CROSS-SECTION AND q<sub>0</sub> < 1



mantle

0

-0.0982

-0.0717

-0.0451

-0.0185

0.0080

Ø.ØØØØØE+ØØ

9.82323E-02 psimaxl 0.00000E+00 btor 2.06889E+00 totcur

btmer

Ø.ØØØØØE+ØØ

8.00062E+05

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0

0.1674

0.1143 0.1409

0.0877

psi

0.0612

0.0346

## COMPARISON OF n = 15 MODE (X)

#### With ballooning transformation



#### Without ballooning transformation

**Higher harmonics** 



#### **Finger like structure**