APPLICATION OF A 2D IDEAL MHD LINEAR STABILITY CODE TO FINITE-AMPLITUDE NONAXISYMMETRIC PERTURBATIONS

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47th American Physical Society Meeting

Division of Plasma Physics

24-28 October 2005

Denver, Co.

GENERAL ATOMICS

3D Non-Axisymmetric Effects are Becoming Increasingly Important in Tokamaks

- **3D perturbations can arise from several sources:**
  - Unintended error fields
  - Imposed external non-axisymmetric coils (intended error fields)
  - Nonlinearly saturated instabilities
- **Plasma response is a key ingredient in determining the consequences in each case:**

  ⇒ **Plasma can amplify or suppress or otherwise modify perturbation !**

- **Key problem is to find this response:**
  - Find the self consistent plasma response
  - Determine the consequences of the perturbation and it's response
    - # Displacement jumps, singular currents, and islands at interior rational surfaces
    - # Rotation drag from resonant or non resonant perturbed fields
  - Mitigate the consequences as needed:
    - # Counter boundary perturbations
    - # Correction coils or trim coils

Tokamaks With Non-Axisymmetric Perturbations Can Be Treated As Either 2D Stability Or 3D Equilibrium Problems

- **Perturbed Equilibrium and Stability Problems Are Closely Related:**

  In general equilibria can be considered as stationary points of the energy functional with the plasma having a fixed specified shape

  – Perturbed equilibria are then stationary points of δW_p with the surface held fixed with a prescribed normal displacement $\boldsymbol{\xi \bullet n}$ or normal magnetic field $\boldsymbol{\delta B \bullet n}$ superimposed on the unperturbed plasma surface

  – Can use tools developed for Tokamak stability **and** 3D stellarator equilibria

- **Use both points of view and tools to answer following questions:**

  – What is the linear response to a given boundary perturbation

  – Conversely give certain specified features of linear response, what boundary perturbations can be used to control them

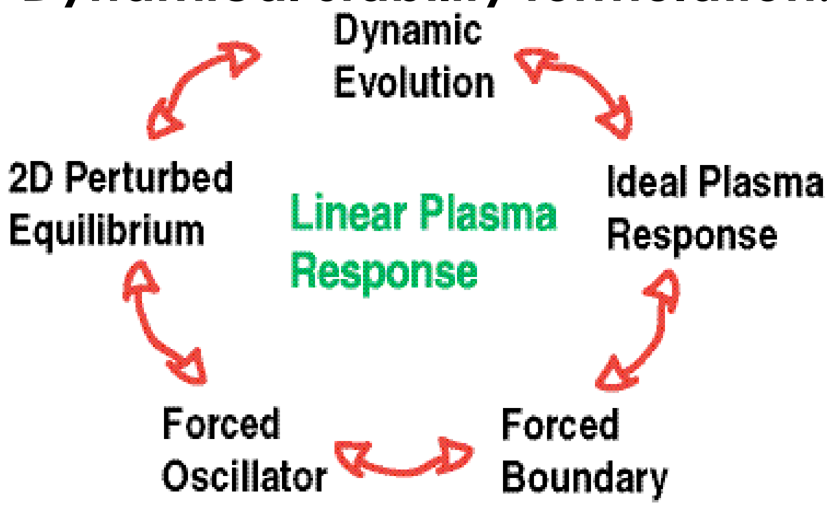
  – What is the nonlinear response to a given boundary perturbation:

  Problem reduces to identifying accessible states and constraints relating initial unperturbed and final nonlinear accessible states

  – What is the relation between the linear and nonlinear responses

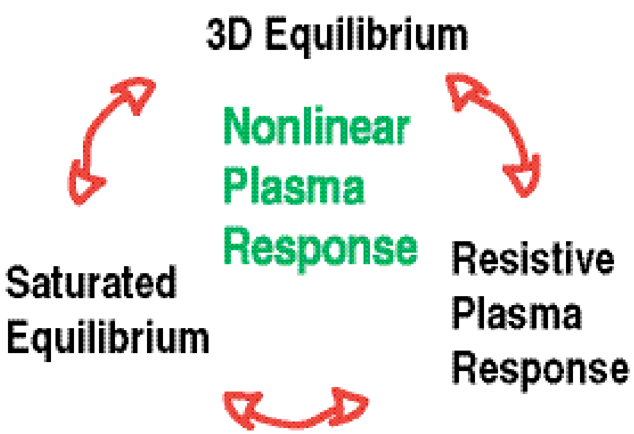
Issues Naturally Fall In Two Distinct Approaches Characterized By 2D Stability And 3D Equilibrium

- **Dynamical stability formulation:**



- **Hilbert Space Formulation**
- **Nuhrenberg-Boozer et al.** (Phys Plasmas 10 2840 (2003))
- **Normal Mode Approach (NMA))** (Chu, et al, Nucl. Fusion 43, 441 (2003))

- **Equilibrium formulation:**



- **Almost Ideal MHD (AIMHD): Jensen** (Phys. Plasmas 8, 5158 (2001).
- **Greens Function Formulation**
- **Response Matrix Formulation**

Focus Separately on Two Distinct Formulations: Linear and Nonlinear Aspects

- **Linear perturbative dynamical system approach:**
  - Compute self-consistent linear response to external boundary perturbations
  - Construct relation between features of linear response and imposed perturbation:
    - ⇒ **Inverse relation obtained by SVD yields set of specific boundary perturbations of 2D equilibrium that control individual (desired) features**
  - Apply to resonant response (Nuhrenberg-Boozer):
  - ⇒ **Set of specific boundary perturbations that control individual displacement jumps**
  - ⇒ **Island widths associated with displacement jumps and boundary perturbations**
  - ⇒ **Optimized shape perturbations to remove critical islands induced by error fields**
  - Apply to nonresonant response: Control nonresonant and resonant features
- **3D saturated state equilibrium approach:**
  - Use instabilities computed for the base 2D equilibrium as a boundary condition to V3FIT to generate a 3-D equilibrium:
    - ⇒ Perturb the base 2D equilibrium using linear nonaxisymmetric eigenmodes from GATO as finite boundary perturbations
    - ⇒ Compute a self-consistent 'force balancing' nonlinear response needed to reproduce a new 3D equilibrium having this boundary

Hilbert Space Approach: Linear Ideal Plasma Response From Projecting Eigenmodes on Plasma Boundary

- **Relate plasma response to boundary perturbations through eigenmodes** (following Nuhrenberg and Boozer (2003) and Chu, et al (2003)):
  - Expand plasma response as a set of complete eigenmodes
  - Project this complete set of the full eigenmodes on to the boundary as a set of boundary displacements
- **These plasma boundary perturbations form a complete set of admissible boundary perturbations under certain conditions:**
  - A boundary perturbation can be expanded as a linear combination of a subset of these perturbations

With important provisos !

- **This yields the complete plasma response as the same linear combination of the corresponding full plasma eigenmodes**

The complete plasma response to any boundary perturbation can be obtained in this way

- **What defines the initial complete set of eigenmodes?**

Under Some Conditions Eigenvectors of Hermitian Operators Form Complete Bases For Hilbert Space of Square Integrable Functions

- **Admissible MHD perturbations: Hilbert space of square integrable functions**
- **The ideal MHD operator $\mathcal{L}$ is Hermitian:**
  - Eigenvalues are real, bounded from below, and countably infinite
  - Eigenvectors are orthogonal
  - If $\mathcal{L}$ were bounded and compact then eigenvectors form a complete set
- **Ideal MHD operator is a convenient choice since:**
  - The eigenvectors are orthogonal in the natural energy norm
  - Eigenvectors of $\mathcal{L}$ represent natural motions so might be expected to minimize the number of terms needed to represent any reasonably interesting perturbation
- **But $\mathcal{L}$ is a non compact (unbounded) operator:**
  - **Completeness of the eigenvectors is non-trivial !**
  - Non square-integrable solutions (continuum modes) exist

These issues need to be resolved before we can use eigenvectors of $\mathcal{L}$

MHD Operator Does Satisfy Conditions for Eigenvectors to Form Complete Bases For Hilbert Space of Admissible Functions

- **The continuum modes must be removed:**
  - Admissible sequences of functions converging to these solutions outside Hilbert space are still approximate solutions
  - **Use the reduced operator $\mathfrak{K}$ obtained by using only normal displacements $X = \xi . \nabla\psi$ to avoid continuum completely**
- **The operators $\mathcal{L}$ and $\mathfrak{K}$ are bounded from below and have a compact inverse:**
  - Compact $\Rightarrow$ bounded sets map to compact sets (I.e. convergent sequences stay inside the set so remain square integrable)
  - Inverse of $\mathfrak{L}$ and $\mathfrak{K}$ is a Greens Function which is compact

  $\Rightarrow$ **Eigenvectors of $\mathcal{L}$ or $\mathfrak{K}$ do form a complete set to represent any admissible MHD displacement normal to flux surfaces**
- **Eigenvectors of $\mathfrak{K}$ are similar to those of $\mathcal{L}$ and have the same convenient properties:**
  - The eigenvectors are orthogonal in the modified energy norm
  - Eigenvectors of $\mathfrak{K}$ are physical motions

Hilbert Space Approach Utilizes Ideal MHD Codes to Construct Full Plasma Response

- **Tabulate complete set of linear perturbation eigenfunctions:**
  - Determine all linear eigenmodes using ideal MHD stability code: ξ_i: $\mathbf{i = 1,N}$
  - Extract normal component $\mathbf{X_i} = \xi_i \cdot \nabla\psi$

$\Rightarrow$ **Complete orthonormal basis: $X_i(\psi,\chi,\phi)$**

- **Project out the boundary displacements:**

$$\mathfrak{X}_i(\chi,\phi) = \mathfrak{P}\, X_i(\psi,\chi,\phi) = X_i(\psi_s,\chi,\phi):$$

$\mathfrak{P}$ is a projection operator from the full Hilbert space to the Hilbert space of boundary displacements $\mathfrak{X}(\chi,\phi)$

  - The $\mathfrak{X}_i(\chi,\phi)$ form a basis for all admissible $\mathfrak{X}(\chi,\phi)$:

$$\Rightarrow \mathfrak{X}(\chi,\phi) = \Sigma\, g_i\, \mathfrak{X}_i(\chi,\phi)$$

- **For any prescribed boundary displacement compute coefficients g_i:**
  - If the $\mathfrak{X}_i$ are a complete set this is possible in principle

The complete plasma response to the boundary displacement

$$\mathfrak{X}(\chi,\phi) = \Sigma\, g_i\, \mathfrak{X}_i(\chi,\phi)$$

is then: $$X(\psi,\chi,\phi) = \Sigma\, g_i\, X_i(\psi,\chi,\phi)$$

Important Reservations Also Apply to The Projection Assumptions Since Projections Are Not Invertible

- **It is possible in principle to expand any boundary perturbation in the $\mathfrak{X}_i$ only if the $\mathfrak{X}_i$ are a complete set**
- **The boundary projection operator $\mathfrak{P}$ is not one-to-one:**
  - Projection operator $\mathfrak{P}: \mathfrak{F}(\Omega_p) \rightarrow \mathfrak{F}(\Gamma_b)$ has a non-vanishing nullspace in $\mathfrak{F}(\Omega_p)$ of internal modes with no corresponding boundary perturbations:
  - Projections of two functions (not in the nullspace) may be degenerate

    $\Rightarrow$The 'dimension' of the Hilbert space of boundary perturbations is smaller than the dimension of the original Hilbert space

    $\Rightarrow$**Something is lost in going back to the original Hilbert space: the degeneracies cannot be inverted**

Completeness of the $\mathfrak{X}_i$ is not guaranteed !

- **Note: the information lost due to degeneracy of the $\mathfrak{X}_k$ with respect to the X_k may be recovered in some cases if the tangential components of the displacement ξ are projected:**
  - But not for internal modes: These are truly degenerate

The Projection Operator $\mathfrak{P}$ Does Offer Possibility of Finding A Complete Basis For Boundary Perturbations

- **Note that he operator $\mathfrak{P}$ projects the function space $\mathfrak{F}(\Omega_p)$:**
  - Not to be confused with projection **P** of the actual plasma domain $\mathbf{\Omega_p}$ on to the surface $\mathbf{\Gamma_b}$
- **The projection $\mathfrak{P}$ can be considered to transform $\mathfrak{F}(\Omega_p)$ to $\mathfrak{F}(\Omega_p)$:**
  - Define an extension of the $\mathfrak{X} \in \mathfrak{F}(\Gamma_b)$ to all of $\mathfrak{F}(\Omega_p)$ as:

$$(\mathfrak{X}(\chi,\phi),\mathfrak{Y}(\chi,\phi)) \equiv (\mathfrak{X}(\psi,\chi,\phi),\mathfrak{Y}(\psi,\chi,\phi))$$

    with $\mathfrak{X}(\psi,\chi,\phi) \equiv \mathbf{X}(\psi_s,\chi,\phi)$ and $\mathfrak{Y}(\psi,\chi,\phi) \equiv \mathbf{Y}(\psi_s,\chi,\phi)$
  - Then the energy norm inner product on $\mathfrak{F}(\Omega_p)$ is still meaningful on $\mathfrak{F}(\Gamma_b)$ when extended in this way
  - The space spanned by the $\mathfrak{X}_k$ is a subset of the full Hilbert space
- **The projection $\mathfrak{P}$ is a bounded Hermitian operator on $\mathfrak{F}(\Omega_p)$:**
  - All eigenvalues of $\mathfrak{P}$ are either zero or 1 and hence real
  - Eigenvectors of $\mathfrak{P}$ could be used as a basis for the space in principle
  - But: the eigenvalues of $\mathfrak{P}$ are degenerate (all unity or zero)

    $\Rightarrow$The $\mathfrak{P}$ eigenvectors of $\mathfrak{P}$ span a 2D space that needs to be orthogonalized using Gram-Schmidt

A Subset of Projected Eigenmodes Can Provide A Basis for the Boundary Perturbations of Most Interest

- **The $\mathfrak{X}_i$ from projection of eigenvectors of $\mathfrak{K}$ are not necessarily orthogonal in the energy norm in the sense of $\mathfrak{P}(X_k)$ extended to the full Hilbert space:**
  - In principle one can define an inner product with weight function **w** st:

$$(\mathfrak{X}_i , \mathfrak{X}_k) = \int_\Gamma \mathfrak{X}^*_i \, \mathfrak{X}_k \, w \, dS = \delta^i_k$$

- **A subspace of the Hilbert space spanned by a linearly independent subset is itself a Hilbert space:**

  ⇒ A subset of projected eigenmodes of $\mathfrak{K}$ can form a basis by taking only those $\mathfrak{X}_k$ that are nondegenerate (ie. $\mathfrak{P}(X_i) \neq \mathfrak{P}(X_k)$):

  i.e. The subspace spanned by the full eigenvectors X_k which yield linearly independent projections can be taken to generate the $\mathfrak{X}_k$

  ⇒ **Boundary perturbations can be expanded as a unique linear combination of these independent perturbations**

  For linearly independent the $\mathfrak{X}_k$ and $X = \sum a_k X_k$ then:

$$\mathfrak{P}(X) = \sum a_k \mathfrak{P}(X_k) = \sum a_k \mathfrak{X}_k \text{ and if also: } \mathfrak{P}(X) = \sum b_k \mathfrak{X}_k \text{ then } b_k = a_k$$

- **The $\mathfrak{X}_k$ are then a complete basis for $\mathfrak{F}(\Gamma_b)$ in the following sense:**
  - If $\exists\, \mathfrak{X}$: $(\mathfrak{X},\mathfrak{X}_k) = 0 \;\forall\, k$ it cannot correspond to a distinct plasma motion
  - Otherwise if it did correspond to a plasma motion then $\exists\, X \in \mathfrak{F}(\Gamma_b)$ st:

$$\mathfrak{P}(X) = \mathfrak{X} \in \mathfrak{F}(\Gamma_b)$$

Boundary Projection Basis Set is Complete in $\mathfrak{F}(\Gamma_b)$ But Does Not Guarantee Full Reconstruction of Plasma Response

- **Boundary perturbations that are excluded are either:**
  - Null (I.e. not normal to the boundary) or :
  - Degenerate having same projection as some other plasma eigenmode
- **Internal modes (the nullspace of $\mathfrak{P}$) must be ignored:**
  - Not a major restriction in practice since purely internal modes are rare:
  - Even the m/n = 1/1 internal kink mode has a finite boundary perturbation if the boundary condition has a wall at infinity
  - Note also that if an internal eigenfunction $\mathbf{X_0} = \xi_0 . \nabla\psi$ from the nullspace of $\mathfrak{P}$ is added to the reconstructed plasma response $\mathbf{X} \rightarrow \mathbf{X} + \mathbf{X_0}$ then by self-adjointness of the ideal MHD operator the Rayleigh Quotient:

$$\delta W(\xi^\dagger+\xi_0^\dagger,\xi+\xi_0) \;/\; \delta K(\xi^\dagger+\xi_0^\dagger,\xi+\xi_0)$$

  is changed only to second order in $|\mathbf{X_0}|/|\mathbf{X}|$
  - But it can contribute to features of the response such as resonant jumps
- **Degenerate modes can be approximated by least square (SVD) fit with the available basis:** $\mathfrak{X}(\chi,\phi) = \Sigma\, g_i\, \mathfrak{X}_i(\chi,\phi)$
  - In practice the Hilbert space is truncated to finite dimension N:
  - $\Rightarrow$ Least square fit for a specified boundary function to truncated basis is appropriate in this case as well

Given Complete Plasma Response One Can Isolate And Study Specific Features of the Response

- **Hilbert space approach allows one to relate specific features of the plasma response to individual boundary displacement basis $\mathfrak{X}_i(\chi,\phi)$:**
  - Enumerate and quantify some specific feature: For example:
    - # Displacement jumps at resonant surfaces
    - # Any feature of the resonant or nonresonant response that can be enumerated and have a quantified value
  - Tabulate feature from plasma response against the $\mathfrak{X}_i(\chi,\phi)$
  - SVD analysis to isolate sensitivity of specific feature to each $\mathfrak{X}_i(\chi,\phi)$
- **Inverse problem of defining specific boundary displacement needed to control specific features can be solved by inverting SVD coefficient matrix:**
  - SVD analysis determines inverse sensitivity
  - Inverse yields boundary perturbations $\mathfrak{X}$ for a prescribed set of characteristics
  - These are given as linear combinations of the $\mathfrak{X}_i(\chi,\phi)$
- **However those features associated with internal modes cannot be identified from this analysis:**

Restriction results from nonvanishing nullspace of projection operator $\mathfrak{P}$!

Nuhrenberg-Boozer Approach Specifically Isolates Sensitivity of Resonant Surfaces to the $\mathfrak{X}_i$

- **An arbitrary boundary perturbation $\mathfrak{X}(\chi,\phi)$ can induce jumps in the normal displacement $X(\psi,\theta,\phi)$ at rational surfaces:**
  - Enumerate j = 1,M rational surfaces and quantify displacement jumps $\mathbf{[X]_i}$ for each basis function i = 1,N
  - Assume M < N (number of basis modes taken in boundary expansion)
  - Relate jumps $[\xi]_i$ to the eigenfunctions $\mathbf{X_i}$ and so to the $\mathfrak{X}_i$: $\Rightarrow [X]_i = \mathfrak{A}\,\mathfrak{X}_i$
  - This defines an M×N matrix $\mathfrak{A}$ whose N columns are the jump values **[X]**

$$([X]_1\ ,\ [X]_2\ ,\dots\ [X]_N) = \mathfrak{A}\,(\mathfrak{X}_1,\ \mathfrak{X}_2,\ \dots\ \mathfrak{X}_N\ ) = \mathfrak{A}\ \mathrm{I} = \mathfrak{A}$$

For any arbitrary prescribed boundary displacement $\mathfrak{X} = \Sigma(g_i\mathfrak{X}_i)$ the jump is:

$$[X] = \Sigma(g_i\,\mathfrak{A}\,\mathfrak{X}_i) = \Sigma(g_i\,[X]_i)$$

- **Perform SVD analysis on $\mathfrak{A}$:** $\mathfrak{A} = \mathfrak{U}\,\mathfrak{D}\,\mathfrak{V}^t$

$\mathfrak{D}$ is a diagonal matrix with: M nonzero eigenvalues and N-M zero eigenvalues

$\mathfrak{U}$ and $\mathfrak{V}$ are orthogonal: $\mathfrak{U}\,\mathfrak{U}^t = \mathfrak{V}\,\mathfrak{V}^t = 1$

  - Invert $\mathfrak{A}$: $\mathfrak{A}^{-1} = \mathfrak{V}\,\mathfrak{D}^{-1}\,\mathfrak{U}^t$ (**1**/$\mathfrak{D}_{ii}$ set to zero if $\mathfrak{D}_{ii}$ ~ **0**)
  - Then $\mathfrak{X} = \mathfrak{A}^{-1}[X]$ defines the boundary perturbation required to set any prescribed set of displacement jumps **[X]**

The Matrix $\mathfrak{A}$ Embodies the Sensitivity of the Resonant Response to External Perturbations

- **This splits out the resonant from the nonresonant response:**
  - Range of $\mathfrak{A}$ is the subspace of those jumps $\mathbf{[X]_i}$ that can be reached by some $\mathfrak{X}_i$: ie, the jumps that can be induced by some boundary displacement:
    - # Dimension of this subspace is the rank of $\mathfrak{A}$ which is the number of nonzero (or significant) $\mathfrak{D}_{ii}$
    - # This is spanned by the columns of $\mathfrak{U}$ corresponding to these same $\mathfrak{D}_{ii}$
  - The nullspace of $\mathfrak{A}$ is the subspace of the $\mathfrak{X}_i$ that is mapped to zero jump
    - # These only produce nonresonant responses
    - # This is spanned by the columns of $\mathfrak{V}$ corresponding to zero (or insignificant) $\mathfrak{D}_{ii}$
- **We focus here on resonant response:**
  - From SVD decomposition we know which boundary perturbations can affect: i.e. augment or heal jumps (islands in a real plasma) at any rational surface

  $\Rightarrow$ We can control internal resonant response by controlling 3D shape perturbations
- **Eigenvalues of SVD matrix $\mathfrak{D}$ determine response to boundary perturbations:**
  - Large eigenvalue $\Rightarrow$ **large jump induced by corresponding basis boundary perturbation**
  - Small eigenvalue $\Rightarrow$ **corresponding basis boundary perturbation has insignificant effect on jump**

Nuhrenberg-Boozer Application in 2D: Need To Distinguish Two Different Views - Base 2D or Perturbed 3D System

- **2D Base system:**
  - Expand in 2D eigenmodes
  - Find complete linear response and determine sensitivity of unperturbed 2D boundary to small 3D perturbations as described
- **Finite but small 3D perturbation of a 2D system:**
  - Given a pre-existing error field or tearing mode:

    ⇒ **Can expand in either 2D eigenmodes or full 3D eigenmodes**
  - These are equivalent however - just different but 'equivalent' bases for the Hilbert space: ⇒**Expansion in 2D eigenmodes is simpler !**
  - Find complete linear response to small 3D perturbations as described
  - This system may have pre-existing jumps from finite 3D perturbation

  ⇒ Determine which additional 3D perturbations can affect these jumps
- **Displacement jumps in ideal theory correspond to islands at respective surfaces in a real system:**
  - **2D Base system:**

    Nuhrenberg-Boozer approach determines boundary perturbations with largest sensitivity to island formation for an axisymmetric equilibrium
  - **Finite but small 3D perturbation of a 2D system:**

    Controlling the jumps effectively controls the size of islands

The SVD Matrix $\mathfrak{A}$ is Also the Key to Controlling Islands Through Controlled External Perturbations

- **An equilibrium with nested surfaces is most sensitive near surfaces that are resonant with the unperturbed equilibrium to external perturbations:**
  - The surfaces easily split ⇒ island opens up
  - Jumps in $\mathbf{X} = \xi.\nabla\psi$ at rational surfaces imply either singular currents (ideal MHD) or islands

    ⇒ **Islands can be controlled by controlling these jumps**
- **Islands are especially sensitive to the boundary perturbations $\mathfrak{X}$ with nonzero eigenvalues:**
  - These are the perturbations that need to be carefully controlled by external coils

  ⇒ The theory can be used to make the jump vanish at any given surface by a suitable choice of the boundary displacement
- **Theory can yield information on actual island size due to error fields:**
  - Island width can be related directly to size of the jump in the ideal theory
  - We have the specific combinations of boundary perturbations (specific sum of the $\mathfrak{X}_i$) that relate the individual jumps $\boldsymbol{\mathfrak{X}} = \boldsymbol{\mathfrak{A}}^{-1}[\mathbf{X}]$

  ⇒ Island size is directly proportional to this perturbation

Nuhrenberg-Boozer Application Can Control Islands Induced by Error Fields With External Trim Coils

- **The boundary perturbations controlling specific islands can also be related to the δB from coil currents needed to make that perturbation by extracting the equilibrium response part:**
  - Find the non-axisymmetric trim fields needed to control those jumps
- **We can then optimize 3D plasma shape using trim (correction) coils:**
  - Find trim coils to provide the desired external perturbations
  - Relate island amplitudes to trim coil currents
  - Find trim coil currents needed to eliminate particular islands

  $\Rightarrow$Trim coil currents needed to eliminate any given jump:

  $\Rightarrow$Control islands from error fields using the DIII-D internal control coils

This theory is basically the same theory as the general formulation developed by Chu and Chance for RWM feedback

Non Resonant Boundary Perturbations Can Also Be Used As Rotation or Tearing Mode Control Tool

- **Experiments in ASDEX suggest nonresonant field perturbations may suppress islands** (Yu, et al., Nucl. Fusion 40, 2031, (2000)):
  - Experiments in DIII-D intended to demonstrate this were inconclusive since the nonresonant fields applied with the C-coil strongly reduced plasma rotation (La Haye, et al, Phys. Plasmas (2002))
- **DIII-D experiments did not take account of the plasma response in applying the nonresonant field - just a nonresonant vacuum field:**
  - Imposed field may not necessarily be the vacuum field
- **Non resonant perturbations can be controlled if plasma response is included:**
  - Form the complete set of eigenmodes $\mathbf{X}_i$ and boundary projections $\mathbf{\chi}_i$ to obtain the full plasma response
- **Enumerate and quantify nonresonant response:**
  - For each basis function $\mathbf{\chi}_i$ enumerate j = 1,M < N:
  - # **Fourier components of eigenmodes and quantify the amplitudes $\{F\}_i$ at some specified surface (e.g. q = 2) for each basis function i = 1,N > M or**
  - # **Rational surfaces and quantify the amplitudes $\{F\}_i$ for some specified nonresonant harmonic (e.g. m = 2)**

Hilbert Space Approach Can Equally Be Applied to Control Non Resonant Components of Response

- **Identify boundary perturbations responsible for controlling specific components:**
  - Relate amplitudes $\{F\}_i$ to the boundary functions $\mathfrak{X}_i$: $\Rightarrow \{F\}_i = \mathfrak{B}\mathfrak{X}_i$
  - The N columns of $\mathfrak{B}$ are the amplitudes $\{F\}_i$

    For any $\mathfrak{X} = \Sigma(g_i\mathfrak{X}_i)$ the amplitudes of the Fourier components at this surface are then:

$$\{F\} = \Sigma(g_i\mathfrak{B}\,\mathfrak{X}_i) = \Sigma(g_i)\{F\}_i$$

- **Perform SVD analysis on $\mathfrak{B}$:** $\mathfrak{B} = \mathfrak{S}\,\mathfrak{E}\,\mathfrak{T}^t$

  $\mathfrak{E}$ is a diagonal matrix with M nonzero eigenvalues and N-M zero eigenvalues

  $\mathfrak{S}$ and $\mathfrak{T}$ are orthogonal and $\mathfrak{B}^{-1} = \mathfrak{T}\,\mathfrak{E}^{-1}\,\mathfrak{S}^t$ (1/$\mathfrak{E}_{ii}$ set to zero if $\mathfrak{E}_{ii} \sim 0$)
  - Then $\mathfrak{X} = \mathfrak{B}^{-1}\{F\}$ defines the boundary perturbation required to set any prescribed set of nonresonant Fourier amplitudes $\{F\}$

GATO Ideal MHD Stability Code Can be Used To Build Table of Eigenmodes of $\mathfrak{L}$

- **GATO solves for eigenmodes using a standard Finite Hybrid Element Galerkin expansion in both radial and poloidal directions:**
  - Expansion in Hybrid (nonconforming) elements and minimizing δW with respect to adjoint node values yields:

$$AX = \lambda BX$$

  - The vector X contains the node values, A represents the potential energy and B represents the kinetic energy or other appropriate norm
  - System is solved for X from which the eigenvector can be reconstructed
- **Thus each eigenvector is an expansion of Finite Elements:**
  - The eigenvectors of $\mathfrak{L}$ or $\mathfrak{K}$ (not the finite element basis) form the basis for the linear plasma response expansion set
- **Several modifications are required before this can be used:**
  - Continuum removal and conversion to true δW code
  - Expansion of Finite element basis set to permit displacement jump discontinuities

Major Modification Required for GATO Involves Conversion to True δW Code

- **The ideal MHD operator $\mathcal{L}$ has two continuous spectra resulting from unboundedness of the resolvent $(\mathcal{L}-\lambda\mathcal{I})^{-1}$ for λ covering certain ranges**
  - An Alfven wave continuum
  - A sound wave continuum

Continuum modes need to be removed from the spectrum for several practical reasons

- **Continuum modes with $\lambda > 0$ lie outside admissible space:**
  - GATO finds the approximate continuum solutions numerically
  - These are the admissible sequences of functions converging to the solutions outside Hilbert space as mesh is refined
- **Continuum modes in principle are internal modes so would be in the nullspace of the boundary projection $\mathfrak{P}$**
- **Eigenmodes of near zero frequency resonate with a static imposed perturbation and must be excluded:**
  - External zero frequency perturbations can resonate with near-zero frequency approximate continuum modes unless excluded from the eigenfunction basis expansion
  - The approximate continuum solutions are only approximately in the nullspace of $\mathfrak{P}$

Conversion of GATO to δW Code Requires Prior Continuum Restabilization and Removal

- **Standard procedure for eliminating continua is by utilizing alternative norm to provide the continuum modes with zero inertia:**
  - Minimizing the Rayleigh quotient $\boldsymbol{\lambda = \delta W / ||\xi||}$ with a norm $||\xi||$ involving the full displacement ξ yields large displacements near $\boldsymbol{\delta W = 0}$
- **Replace kinetic energy normalization in $\mathcal{L}$ by a more suitable norm:**
  - Use $\mathfrak{K}$: i.e. a $\boldsymbol{\delta K}$ norm utilizing only the $\mathbf{X} = \xi.\nabla\psi$ component of displacement
  - Norm with $\xi \bullet \mathbf{n}$ or $\boldsymbol{\delta B \bullet n}$ specified at a boundary point
  - Norm with surface average of $\xi \bullet \mathbf{n}$ or $\boldsymbol{\delta B \bullet n}$ ("boundary inertia")
  - Note that problem remains Hermitian, bounded from below and has a compact (Greens Function) inverse:
    - $\Rightarrow$ Eigenvectors of reduced problem are still a complete set
- **But: Continuum in GATO is numerically destabilized:**
- **Continuum modes need to be restabilized before applying δW norm:**
  - Sound continuum is easily eliminated by insisting on incompressibility
  - δW norm to eliminate Alfven continuum produces spectral pollution:
  - $\Rightarrow$ Sequence of spurious unstable modes increasing in number with mesh
  - Manifestation of providing zero inertia to numerically destabilized modes

Continuum Removal is Being Done in Several Steps Following Degtyarev, et al.

- **Subtraction of numerical destabilization term from δW:**
  - Subtract numerical term

$$\delta W_c = -\frac{1}{2}\int\left[\frac{h_\psi^2}{\left\langle |B|^2 / |\nabla\psi|^2 \right\rangle}\left(1/8\right)\left[2\langle S\rangle\langle T\rangle + \langle S\rangle^2\right]\left(\frac{\partial \xi_\psi}{\partial\psi}\right) d\psi\right]$$

Here $S = \left(\frac{B\times\nabla\psi}{|\nabla\psi|^2}\right)\bullet\left[\nabla\times\left(\frac{B\times\nabla\psi}{|\nabla\psi|^2}\right)\right]$ is the local shear $T = \frac{\mathbf{j}\cdot\mathbf{B}}{|\nabla\psi|^2} - S$

and $\langle\ \rangle$ represents the flux surface average

  - Force density profile to vanish at the magnetic axis: $\rho \sim |\nabla\psi|^2$
    ⇒ Eliminate spurious modes
  - Replace kinetic energy norm by: $\frac{1}{2\mu_0}\int\left[\rho\left(|\xi\cdot\nabla\psi|^2 / |\nabla\psi|^2\right) J\, d\psi\, d\chi\, d\phi\right]$ $\quad (J \equiv |\nabla\psi\cdot\nabla\chi\times\nabla\phi|^{-1})$

- **Local Shear term can be evaluated in straight field line coordinates:**

$$S = -q'J - q^2(\nabla\chi - \beta\nabla\psi)\cdot\nabla\beta \quad \text{where } \beta = \left(\nabla\psi\cdot\nabla\chi / |\nabla\psi|^2\right)$$

Modifications Also Require Extended Linear Displacement Basis Set

- **Ideal MHD operator $\mathfrak{L}$ or $\mathfrak{K}$ has two linearly independent solutions with different asymptotic behaviour at rational surface singularities:**
  - Small Frobenius solution is the physical (Hilbert space) solution
  - Large Frobenius solution is non-integrable but
  - Existence of large solution permits jumps in small solution displacement
- **Displacements with jump discontinuities at rational surfaces need to be represented by the Finite element basis:**
  - GATO and other ideal MHD codes solve for small solution
  - Still admissible as having finite energy norm
  - Finite Hybrid Elements already allow piecewise constant elements
  - Corresponding infinite displacement derivatives need to be excluded in usual nonconforming element manner
- **GATO uses Finite Hybrid Elements discontinuous at element boundaries in addition to continuous piecewise linear elements:**
  - Key is to ensure infinite derivatives do not contribute to energy
  - This is already handled by hybrid method: Continuous elements used for derivatives equal discontinuous elements only on average

Finite Hybrid Element Method Utilizes Discontinuous Elements for X Tied to Continuous Elements for $\partial X/\partial\psi$

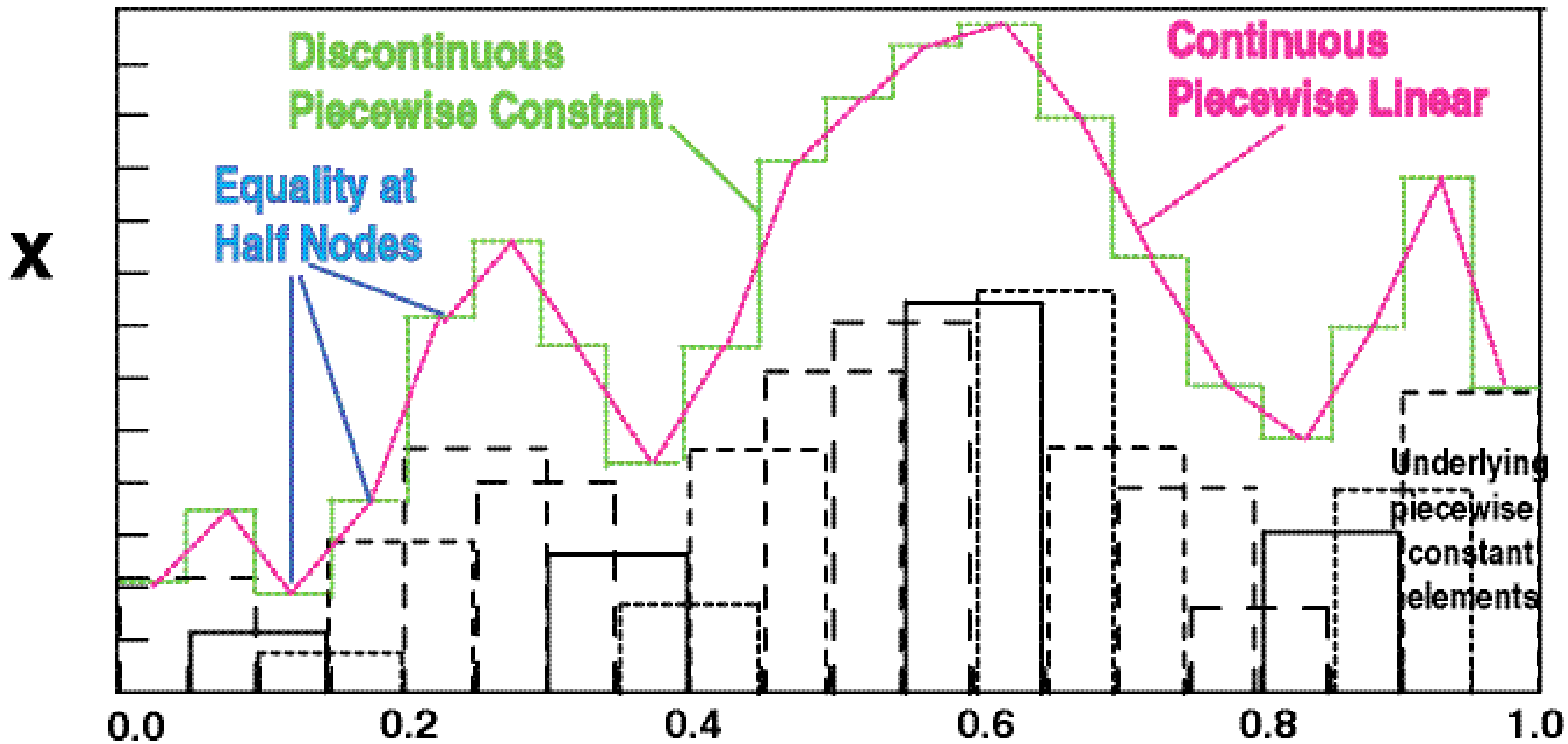
- **Finite Hybrid Element method requires all element functions entering δW to be of same polynomial order:**
  - Displacements $\mathbf{X}^{(1)}(\psi,\chi)$ constructed across two elements centered on single nodes
  - Displacement derivatives $\partial\mathbf{X}^{(2)}/\partial\psi(\psi,\chi)$ utilize tent elements across two elements centered on each node
  - $\Rightarrow$ Construction enables exact equality of key relations such as $\nabla.\xi=0$ everywhere
- **Resonant part of $\partial X^{(2)}/\partial\psi(\psi,\chi)$ always appears with factor (m-nq) which vanishes on resonant surface:**
  - $\Rightarrow$ Contributions from rational surfaces should automatically vanish in the limit as the mesh is refined
  - It may be necessary however to actively suppress the resonant part of $\partial\mathbf{X}^{(2)}/\partial\psi(\psi,\chi)$ in practice

Discontinuous Elements for X Constrained In Natural Way to The Continuous Elements for $\partial X/\partial\psi$

- Element functions for $X^{(1)}(\psi,\chi)$ and $\partial X^{(2)}/\partial\psi(\psi,\chi)$ constructed to have $\int X^{(1)}(\psi,\chi)d\psi d\chi \equiv \int X^{(2)}(\psi,\chi)d\psi d\chi$

  over each cell:

  $\Rightarrow$ $X^{(1)}(\psi_*,\chi_*) \equiv \int X^{(2)}(\psi_*,\chi_*)$ at the half node points $\psi=\psi_*$ and $\chi=\chi_*$



Can reconstruct solution from either continuous or discontinuous elements !

3D Equilibrium Approach Can Study Nonlinear Response

- **Assume plasma is perturbed by a fixed specified displacement:**
  - Previous approach will describe linear plasma response:
    - $\Rightarrow$ This response is still in equilibrium to first order
    - $\Rightarrow$ Will it evolve dynamically to a new nearby state due to unbalanced forces second order in the linear displacement?
- **In quasilinear regime the displacement modifies base equilibrium to become a 3D system with finite perturbation:**
  - This will continue to evolve dynamically:

    Nonlinearly unstable $\Rightarrow$ no final state

    Nonlinearly stable $\Rightarrow$ final saturated stationary 3D equilibrium state
- **Recompute this new state as a 3D equilibrium**

Constraints Imposed In Equilibrium Construction Not Clearly Related To Those Imposed During Actual Dynamic Evolution

- **New 3D Equilibrium Evolves Under Certain Constraints: (AIMHD)**
- **Fixed (or a subset of points on) boundary specified :**
  - $\Rightarrow$ Assumes plasma response does not change boundary
  - $\Rightarrow$ Equivalently the boundary is forced
- **Specific assumptions made concerning topology:**
  - – VMEC equilibria assume nested flux surfaces
  - – PIES or HINST equilibria do not assume nested surfaces but:
  - # Assumptions made in PIES about profiles within islands (see below)
  - # Assumptions made in HINST about dynamics of evolution (inertia etc.)
- **Specific assumptions made concerning profile constraints:**
  - – Generally p = p(ψ,region), where region is a simply connected region isolated from other regions by a separatrix:
    - $\Rightarrow$ If new regions open up assumptions need to be imposed on p(ψ,region) for those regions
  - – Current density in 3D non-nested codes is set by an arbitrary flux dependent integration constant

These Questions Are Essentially Those Posed by The Theory of Almost Ideal MHD (AIMHD)

- **Obvious links between this and the theory of AIMHD:**
  - What are the constraints actually being imposed in the 3D equilibrium calculation?
  - What relation do they bear to the actual physical constraints imposed by the physical system during the dynamics of equilibrating to a new 3D saturated state?
- **For now we take the approach that:**
  - The boundary is treated as a forced boundary:
    - $\Rightarrow$ i.e. constrained by either external fields or that of a linear ideal MHD eigenmode
  - The topology remains nested
  - The profiles remain fixed as functions of normalized poloidal flux:
    - $\Rightarrow$ i.e. $p = p(\psi)$ is fixed

But other assumptions could be made instead!

Formulation for Linear and Nonlinear Plasma Response to External Perturbations Has Applications To Tokamaks

- **Hilbert space formulation (and variants) provides linear response:**
  - Two key formal theoretical problems resolved satisfactorily:
    - Completeness of eigenvectors of $\mathfrak{L}$
    - Non invertibility of the boundary projection operator $\mathfrak{P}$
  - Two key practical problems resolved satisfactorily:
    - Continuum removal is necessary and can be implemented
    - Extension of basis to include displacement jumps

  Applied to resonant response $\Rightarrow$Control of islands

  Applied to nonresonant response $\Rightarrow$Possible control of tearing modes
- **Equilibrium formulation (and variants) provides nonlinear response:**
  - Example of Almost Ideal MHD formulation
  - Specific assumptions of constraints justifiable in principle but alternative assumptions are possible
- **Linear and nonlinear response formulations intimately related:**
  - Greens Function and Response Matrix formulations are essentially inverse problem to Hilbert space differential operator formulation
  - Equilibrium and stability are related: **Perturbed equilibria are stationary points of δW_p with the surface prescribed by a normal displacement**