Implementation of a new Algorithm for Linear Resistive MHD Stability

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Forty-Sixth Annual Meeting
Division of Plasma Physics
November 15- 19, 2004, Savannah, Georgia

Outline

Introduction

A method for evaluating linear resistive MHD stability [1] has been implemented for the 2D toroidal case in the TWIST-R code. In 1D, the algorithm was previously shown to be accurate, with good convergence, even for irregular singularities [2]. Challenging numerical issues arise in 2D due to coupling of the additional regular solution components, which need to be treated properly across the singular layer, singularities in coefficients that need to be cancelled exactly, and coordinate singularities at the magnetic axis. Using an analytic Solov'ev equilibrium to systematically check and benchmark the code, all the major numerical issues have been resolved and physically meaningful solutions and asymptotic matching data are obtained. Convergence in most cases is near quadratic for a range of values of the Mercier index $\mu = \sqrt{-D_i}$ well beyond that possible for previous asymptotic methods. Benchmarks for the Solov'ev case and numerically generated equilibria will be discussed.

TEARING STABILITY IS AN UNRESOLVED THEORETICAL PROBLEM

- Theory of tearing modes in low β I-D geometry is well established
  - Linear: Sign of $\Delta' \propto [\Psi']/\Psi \propto C_S/C_L$ provides stability criterion (FKR)
    $\Delta' = \Delta_{innerlayer}$ (Q) provides growth rates (GGJ)
  - Nonlinear $\frac{dW}{dt}$ describes island growth (Rutherford)
- Extension to finite β 2-D systems has been problematical
  - Rational surfaces are coupled Δ' is not a number!
  - At finite β, $[\Psi']/\Psi$ but C_S/C_L is finite for $1/2 < \mu < 1$:
    How is either related to linear or nonlinear stability?
  - Neoclassical effects important in nonlinear regime and connection between linear, Rutherford, and neoclassical regimes is not fully clear
- Many issues were resolved by Chu, Pletzer, Dewar, Greene (1993) but some still remain:
  - Numerical issues not fully resolved for $\mu > 1$
  - $\Delta' = C_S/C_L$ is finite for $\mu < 1$ but C_S/C_L -> infinity at integer μ

New approach fully resolves numerical issues and has potential to resolve remaining physics issues

TWIST-R FORMULATION HAS SIGNIFICANT ADVANTAGES OVER OTHER RESISTIVE MHD STABILITY CODES

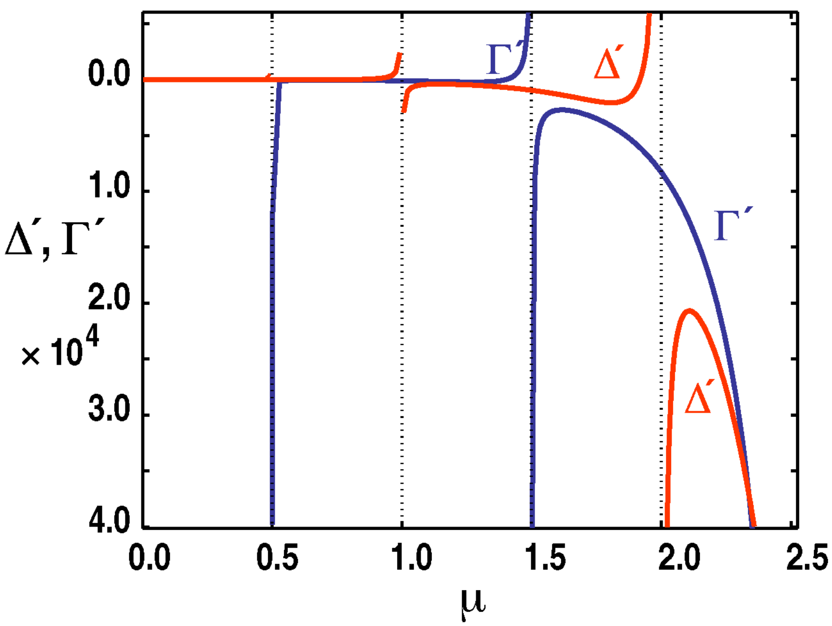
- Asymptotic matching codes have significant advantages for linear calculations over more comprehensive resistive MHD codes MARS, NIMROD or M3D:
  - High S limit applicable to experiments ($S > 10^7$)
    (cf $S \sim 10^4$ to 10^5 for initial value codes)
  - Fast and accurate (cf hours for linear runs of initial value codes)
  - Suitable for exploring detailed physics issues and for extensive parameter scans
- TWIST-R is an asymptotic linear resistive MHD matching code, similar in principle to PEST-III but without well known PEST-III limitations:
  - No restriction on μ and therefore β or profiles (cf PEST-III: $1/2 \le \mu < 1$)
  - Numerically stable for infinite mesh (cf PEST-III: in principle unstable with very fine meshes)
  - No restriction on inner layer physics model (cf PEST-III: GGJ model)

TWIST-R formulation is a new general algorithm for solving differential equations with singular solutions

$\Rightarrow$ General applications outside fusion and plasma physics

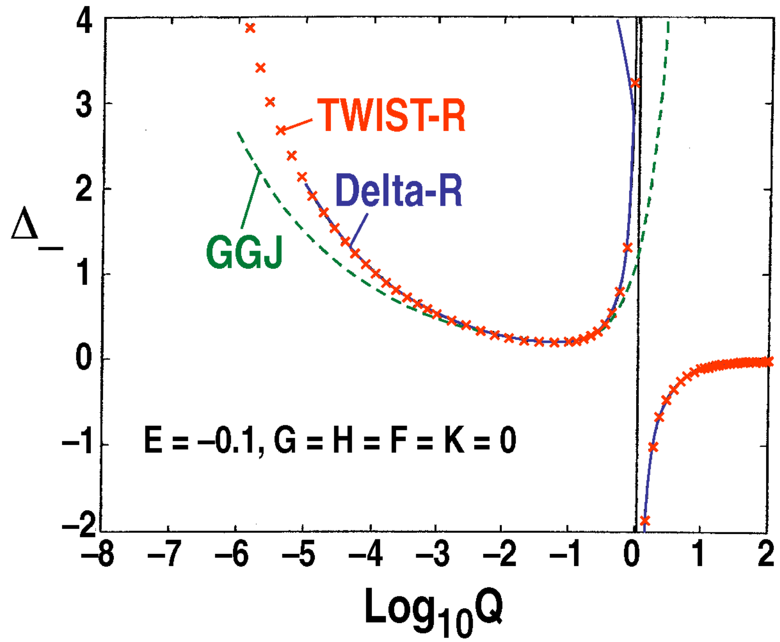
TWIST-R CODE WORKS EXTREMELY WELL FOR 1-D PROBLEM

- New formulation in TWIST-R code transforms asymptotically singular solutions to finite solutions
  - Valid for arbitrary $\mu = \sqrt{-D_I}$ ($D_I = -1/4 + p'/q'^2\, V^{\dagger\dagger}$) (cf PEST–III valid for $1/2 \le \mu < 1$)
    - $\Rightarrow$ Valid for arbitrary β
- The approach was tested on one dimensional Sturm-Liouville problem with one singular point, which models the Newcomb equation with a resonance harmonic around a single rational surface:
  - Comparison between the results from this approach with the known analytic solution has shown excellent accuracy and robustness of this method
- The formulation was shown to also work for arbitrary β using a specially constructed analytic cylindrical equilibrium sequence
  - Reproduces well the parity selection poles at integer and half integer μ even beyond $\mu = 1$



TWIST-R FORMULATION ALSO WORKS EXTREMELY WELL FOR 1-D SIXTH ORDER INNER LAYER PROBLEM

- The technique can also be applied to the resistive inner layer problem with irregular singular points.
  - Inner layer problem is sixth order with irregular singularity:
    - ⇒ Additional non Frobenius solutions present
  - Additional large exponential solutions eliminated from boundary conditions
  - Additional small exponential solutions are treated but do not invalidate the technique



- Results reproduce previous published results over large parameter range

  (Glasser et al, Phys. Fluids, 27, 1225, (1984))

- Extendable to other inner layer models:
  - Density profile model

    (Galkin, et al, Phys. Plasmas, 9, 3969, (2002))

NEW FORMULATION FOR LINEAR TEARING STABILITY IS IMPLEMENTED FOR ARBITRARY β NONCIRCULAR 2-D GEOMETRY

- 2-D version is implemented numerically as finite difference scheme in radial direction and spectral in poloidal direction
- Additional complications arise in 2-D:
  - — Nonresonant poloidal harmonics are present and are coupled to the resonant large and small Frobenius components
  - — Nonresonant poloidal harmonics generate a regular component of the resonant harmonic
    - $\Rightarrow$ Regular solution is continuous across rational surface
    - $\Rightarrow$ Small Frobenius solution is subdominant to regular solution harmonics
  - — Spectral pollution needs to be avoided in finite difference scheme
- 2-D code is undergoing testing and benchmarking
  - — Tested extensively for analytic Solovev equilibrium

$\Rightarrow$ Numerically calculated metric coefficients can be compared term by term to analytic coefficients

Solov'ev equilibria

Grad-Shafranov equation with a given toroidal current j_ϕ:

$$r\Delta^* \psi(r,z) = -rp' - \frac{1}{r} ff' = j_\phi \tag{1}$$

has a family of analytical solutions [Solov'ev,1968], managed by a few parameters. Consider p' and ff' distributions:

$$\begin{aligned} \frac{dp}{d\psi} &= 8(1+\alpha^2)C_0; \\ \frac{df^2}{d\psi} &= -16\alpha^2\sigma^2 C_0; \end{aligned} \tag{2}$$

then the solution is

$$\psi(r,z) = C_0\left[(\Delta^2 - r^2)(r^2 - \delta^2) - 4\alpha^2(r^2 - \sigma^2)z^2\right]; \tag{3}$$

where parameters Δ, δ and magnetic axes coordinates:

$$\Delta = \max r, \quad \delta = \min r, \quad (r_m, z_m) = (\sqrt{\frac{\Delta^2+\delta^2}{2}}, 0) \tag{4}$$

Major plasma radius R and aspect ratio A are correspondingly:

$$R = \frac{1}{2}(\Delta + \delta); \quad A = \frac{\Delta + \delta}{\Delta - \delta}; \tag{5}$$

Magnetic surfaces $\psi(r, z) = \text{const}$ are given by parametric formulas:

$$r^2 = \frac{\Delta^2 + \delta^2}{2} \left(1 + \omega(a) \cos \Theta\right) \ , \tag{6}$$

$$z\sqrt{r^2 - \sigma^2} = \frac{\Delta^2 + \delta^2}{4\alpha} \omega(a) \sin \Theta \ , \tag{7}$$

where

$$\omega(a) = \frac{\Delta^2 - \delta^2}{\Delta^2 + \delta^2} \sqrt{a} \ , \tag{8}$$

and a is a magnetic surfaces label and Θ is an periodic coordinate:

$$a = 1 - \psi/\psi_m \in [0, 1] \ , \quad \Theta \in [0, 2\pi] \ . \tag{9}$$

Δ' and Γ' Calculations

Analytical Solov'ev equilibria were widely used for both code testing and to compute $\Delta'(\mu)$ and $\Gamma'(\mu)$ dependencies. This has the unique advantage that it is a fully toroidal analytic solution for which the metric coefficients and Jacobian can be derived analytically and used for comparison. The detailed description of the analytic equilibrium, including metric coefficients representation, Jacobian, coordinate transform etc., is given in Ref.[3]. A typical toroidal equilibrium configuration with aspect ration $A = 3$, parameters $\Delta = 2, \delta = 1, \alpha = 1.0005, \sigma = 0.05$ and corresponding profiles $p\prime, ff', q$ is presented in Fig.1 on grid $N \times M = 64 \times 128$. There is only $m/n = 2/1$ $(q = 2)$ resonance surface for the case, which is marked with red dashed line. Mercier index at the resonance surface is $\mu = 1.39$.

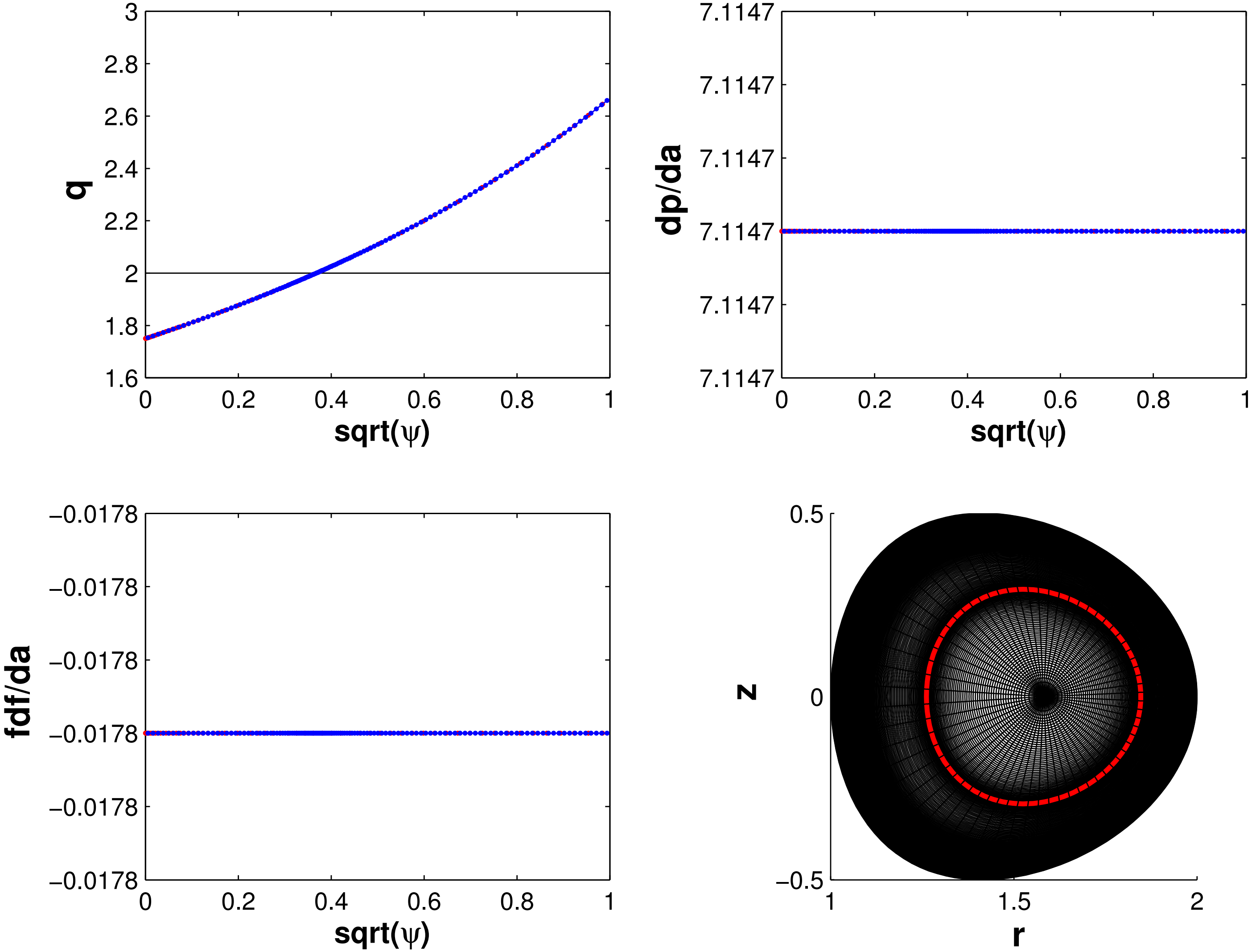


Figure 1: Solov'ev equilibrium and profiles

Eigenfunctions and its harmonics

The TWIST-R code computes total solution (eigenfunction) of Euler equations for inertia free equilibrium plasma configuration and then estimates Frobenius expansion coefficients for large, small and regular solutions, which compose the total solution [1]. The computed total solution (eigenfunction), scaled with $x^{0.5+\mu}$, for the presented above equilibrium Fig.1 is shown in Fig.2. Here $x = |\psi - \psi_r|/\psi_{max}$ is the distance from the resonance surface labelled with ψ_r. Toroidal angle cross section is displayed here and the inner red circle marks the resonance surface location.

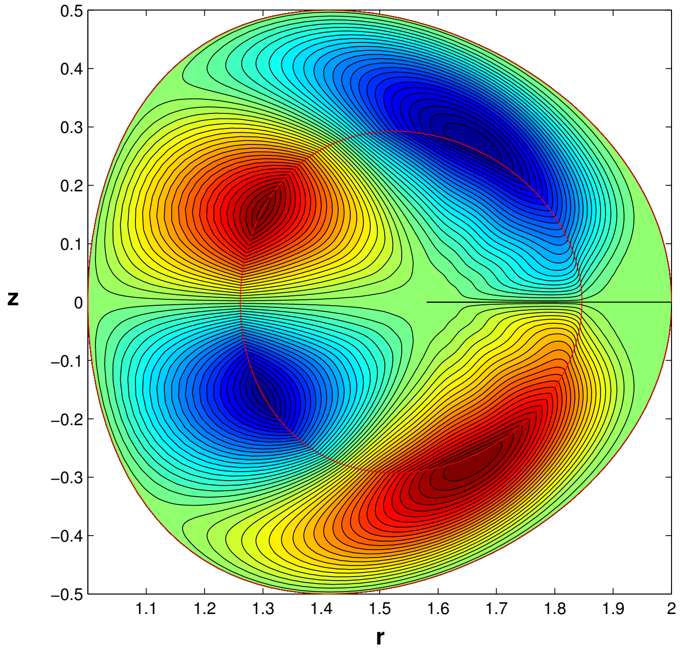


Figure 2: Total solution (scaled displacement) for Solov'ev equilibrium with $\mu = 1.39$

11 poloidal harmonics were involved in the calculation and the radial distribution of first six of them (the rest even smaller) is shown in Fig.3.

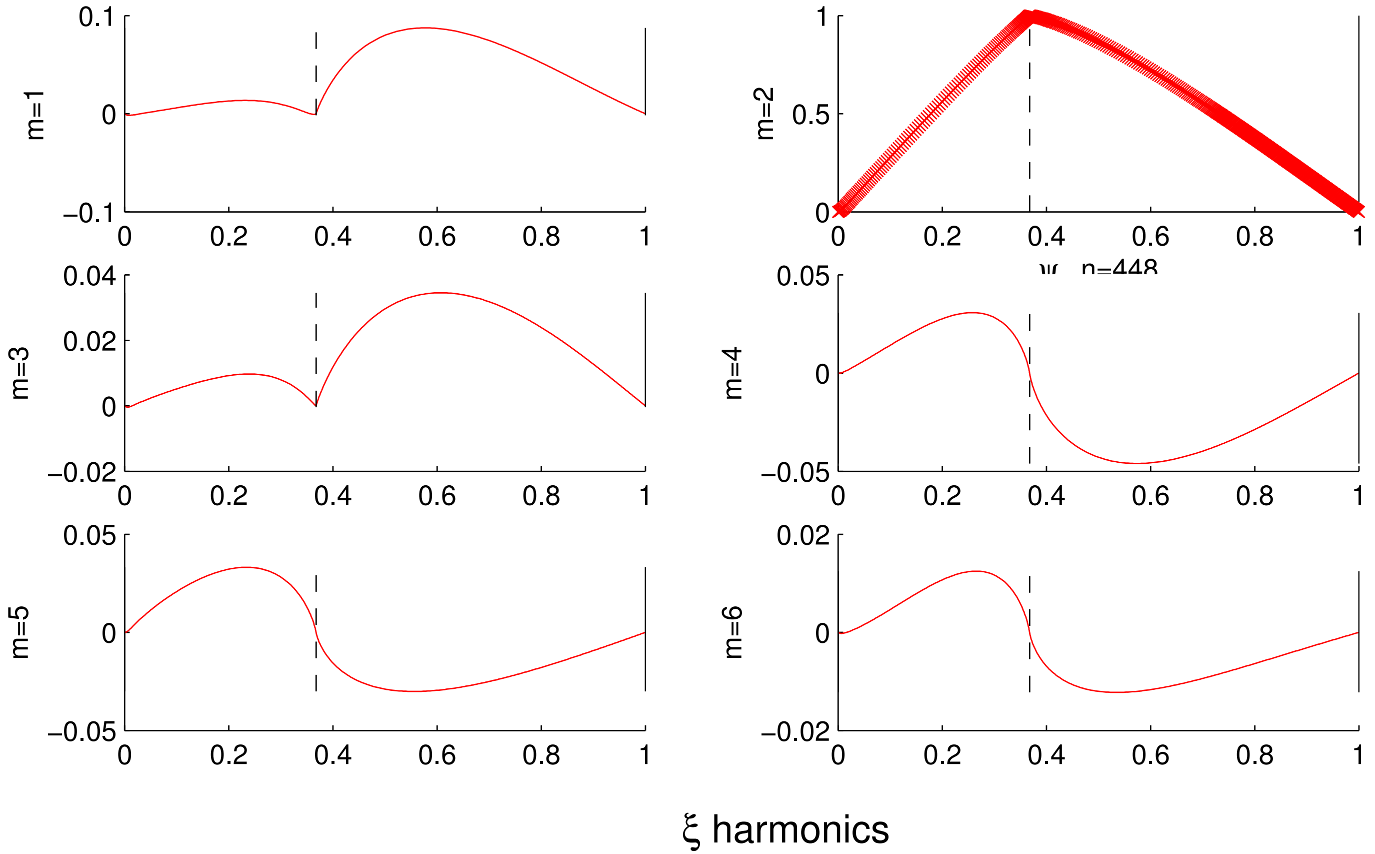


Figure 3: Poloidal harmonics for the radial component of the total displacement $\tilde{\xi}$ with $\mu = 1.39$

For another Solov'ev equilibrium with $\mu = 0.82$, the total displacement is shown in Fig.4

Solovev equilibrium: 2/1 mode: μ = 0.82 chosen for benchmarking

Contours of $\tilde{\xi}$ **Normal and tangential m=2 components:**

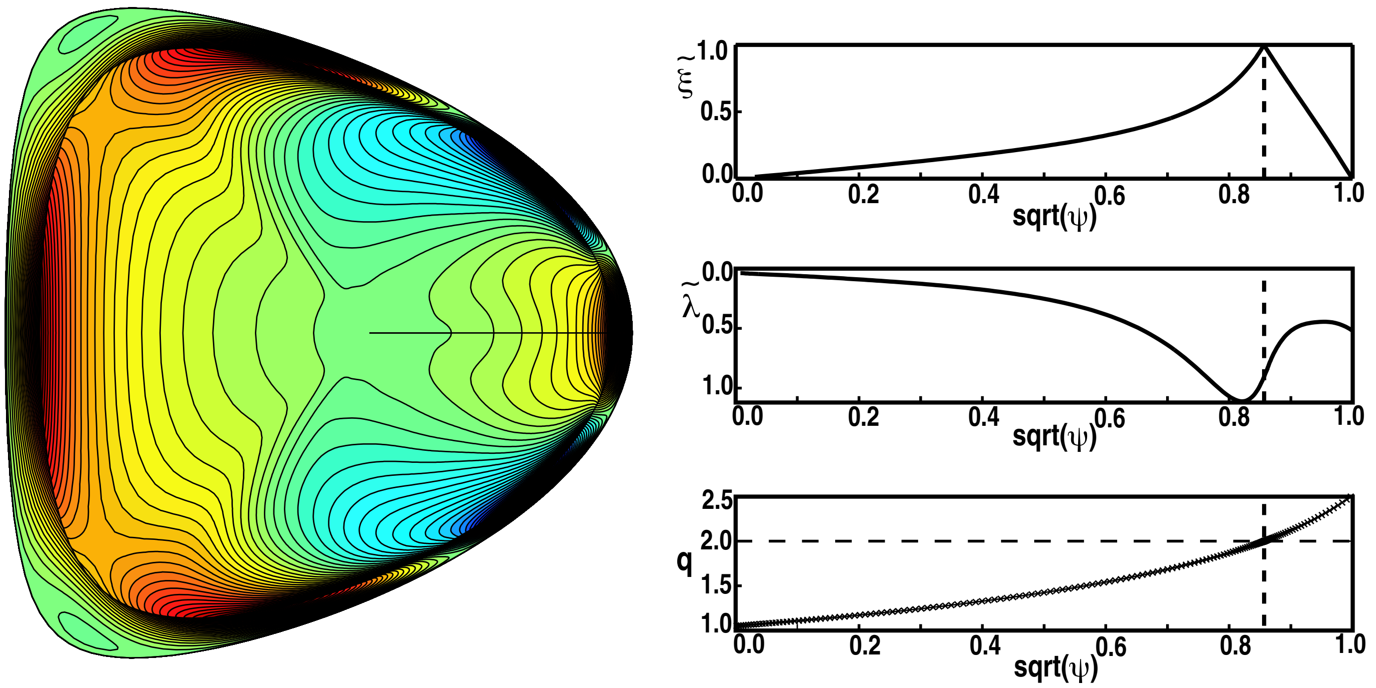
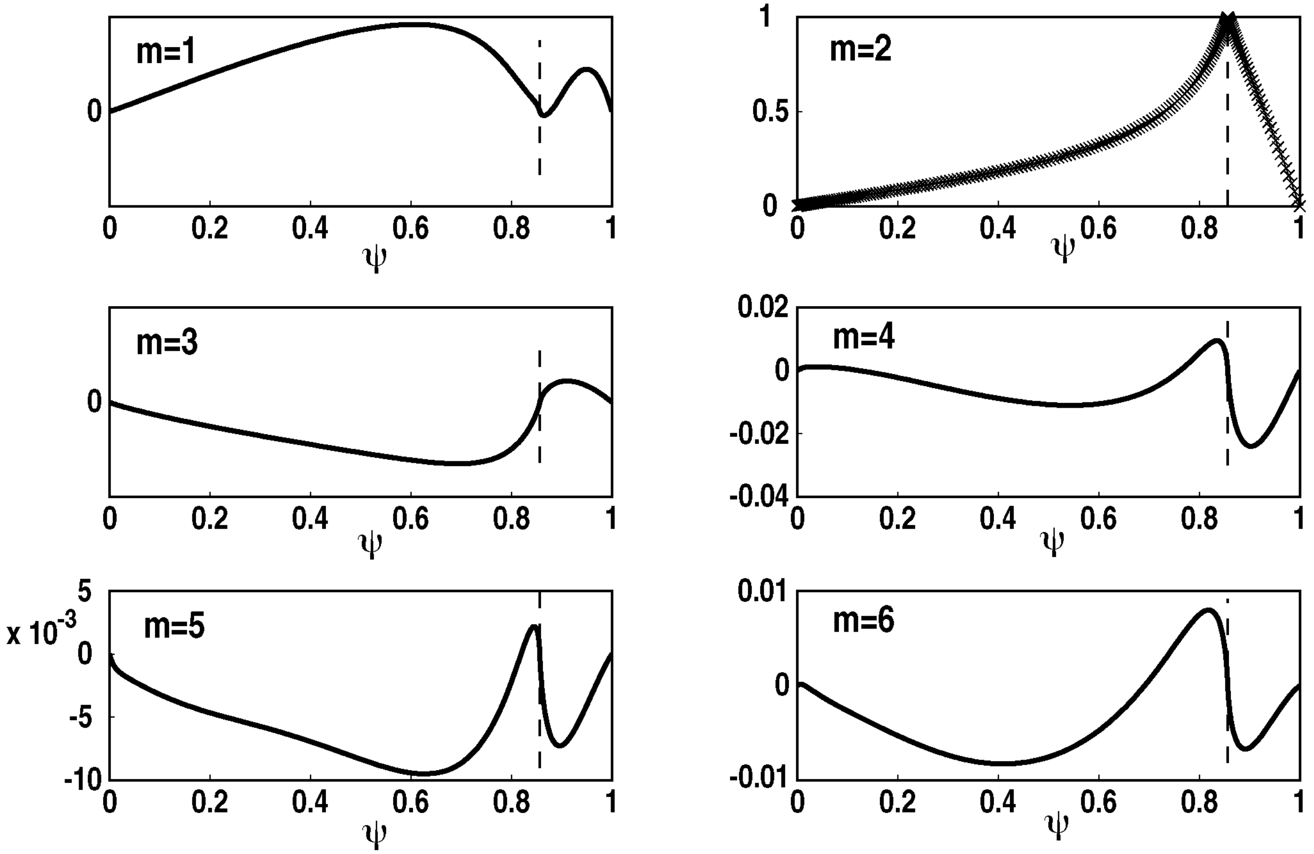


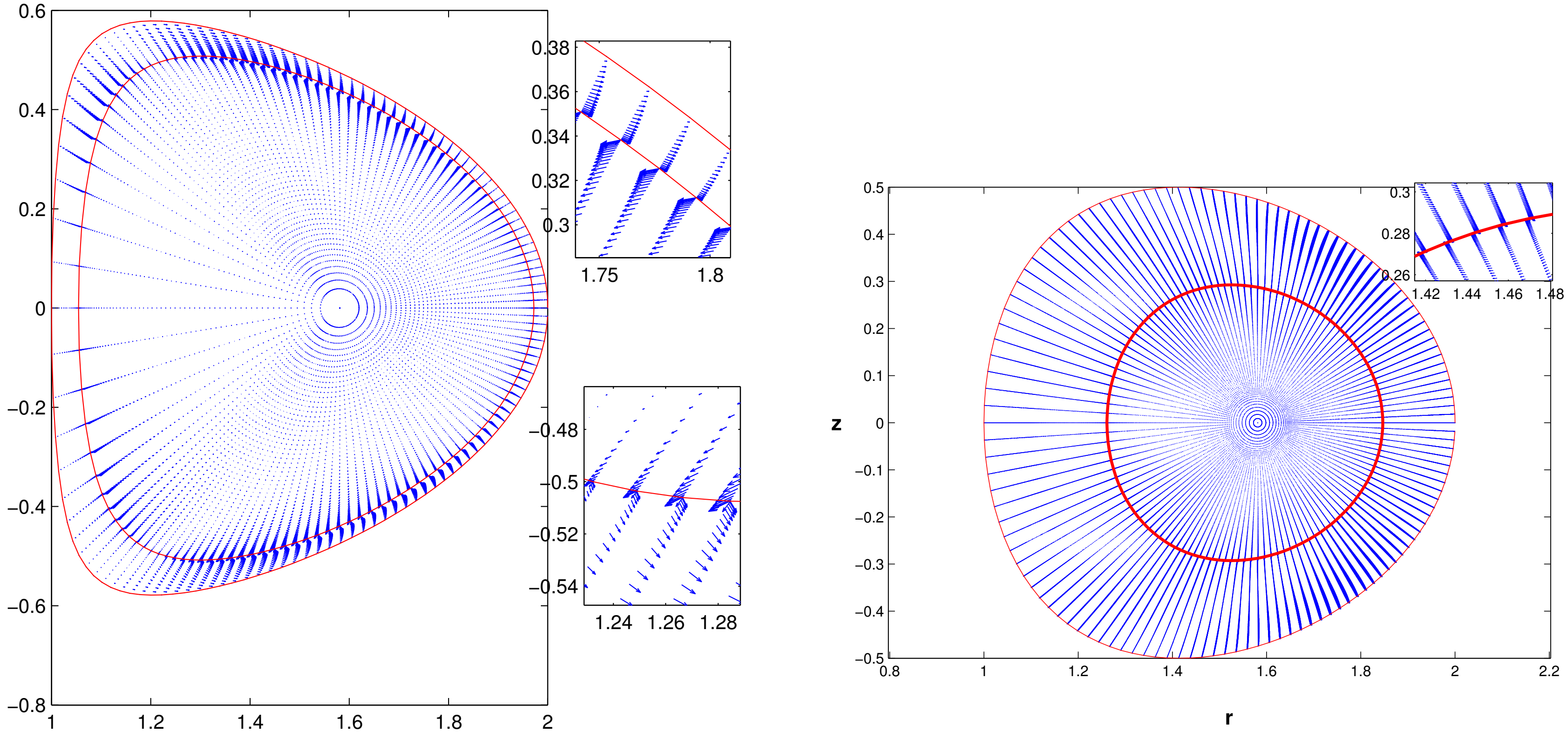
Figure 4: Total scaled displacement for Solov'ev equilibrium with $\mu =$ 0.82

Vector plot of the scaled displacement shows a tearing structure of the solution for both cases, see Fig.5.

NUMERICAL SOLUTIONS FOR μ = 0.82 SHOW PHYSICALLY EXPECTED BEHAVIOR OF NON-RESONANT HARMONICS

ξ harmonics





(a) $\mu = 0.82$ (b) $\mu = 1.39$

Figure 5: Tearing mode structure of the scaled eigenfunction for Solov'ev equilibrium

Convergence

Radial grid number N and poloidal harmonics number M were varied to study the convergence. Typical behavior of eigenfunctions with different N number is shown in Fig.6.

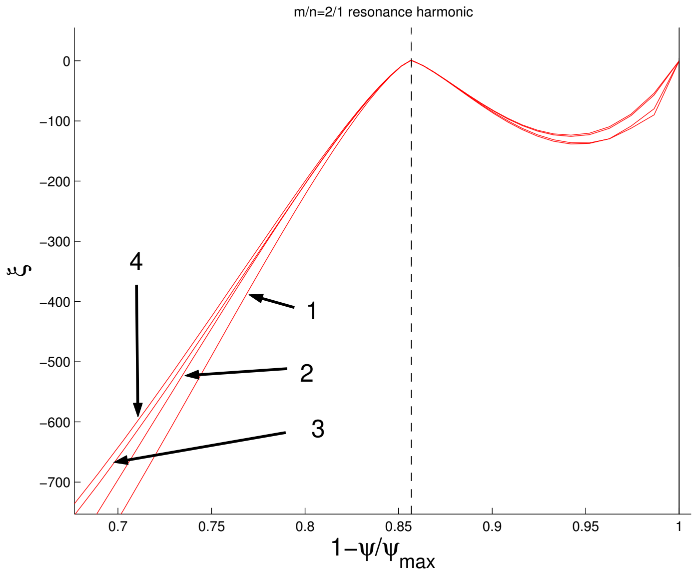


Figure 6: Eigenfunction behavior near resonance surface with N = 64, 128, 192, 256

A best fit data approximation is used to analyze the convergence rate automatically from the computed data with different grid numbers. The convergence behavior is given in Fig.7. The estimated convergence rates were found to be -1.827 for Δ' (close to quadratic) and -1.27 for Γ' (they are also displayed in x-axes figure labels) and are used to get the converged values Δ'_∞ and Γ'_∞ (which corresponds $n = \infty$).

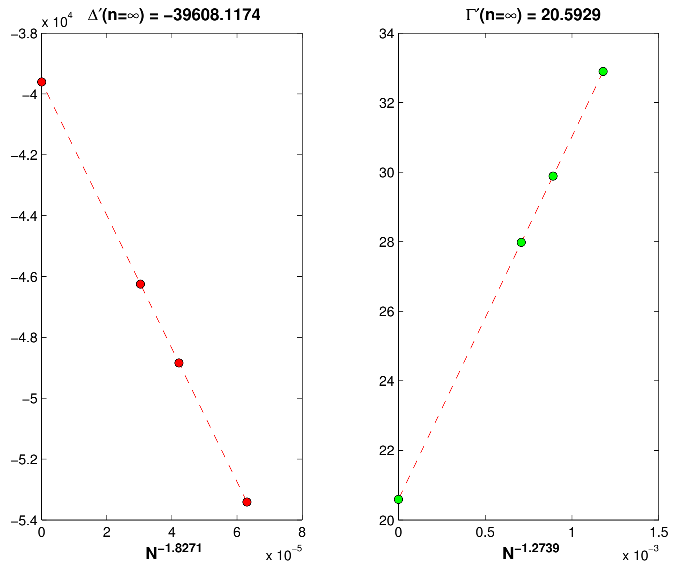


Figure 7: Convergence of Δ' and Γ' for $\mu = 1.39$

$\Delta'(\mu)$ and $\Gamma'(\mu)$ for Solov'ev equilibria

Varying parameters $\sigma, q(0), \alpha$ (see Sec. II A), equilibria with the Mercier index μ in the range $0.5 < \mu < 1.5$ was obtained. Monotonic q profile with $1 < q_{min} <= q <= q_{max} < 3$ is used to work with one resonance surface only. All equilibria should be ideal MHD stable, what was checked with ideal MHD stability code TWIST [8, 9]. A few computed $\Delta'(\mu)$ and $\Gamma'(\mu)$ for Solov'ev equilibria are shown in Fig.8.

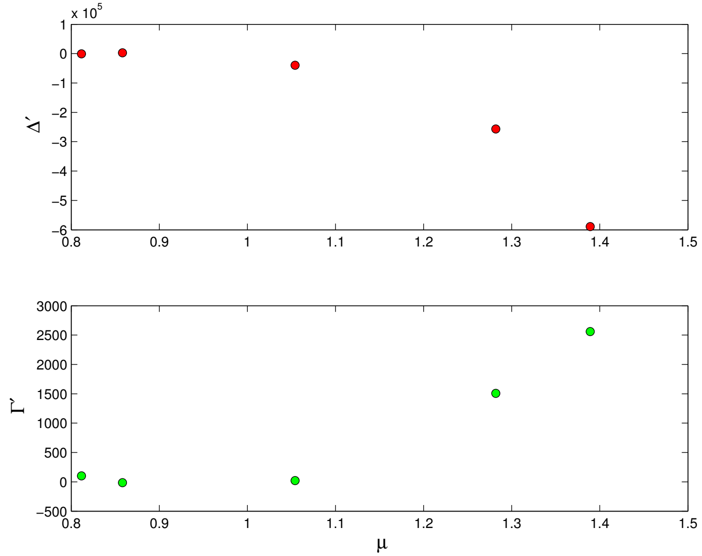


Figure 8: $\Delta'(\mu)$ and $\Gamma'(\mu)$ for Solov'ev equilibria

Numerically computed equilibria

Numerically computed by POLAR-2D code [10], equilibrium series was analyzed against resistive modes. The equilibrium family was prescribed by given distributions of the plasma current with profiles:

$$\begin{aligned} \frac{dp}{d\psi} &= p_0 \frac{\epsilon_1 \epsilon_2}{\psi_{max}} (1 - \epsilon_2 a)^{\epsilon_1 - 1}; \\ \frac{df^2}{d\psi} &= f_0^2 \left(|\psi_{ma} - \psi_{edge}|^{\alpha_1} - |\psi_{ma} - \psi|^{\alpha_1} \right)^{\alpha_2}; \end{aligned} \tag{10}$$

where corresponding plasma pressure was $p(\psi) = p_0(1 - \epsilon_2 a)^{\epsilon_1}$ and ψ_{ma} and ψ_{edge} were poloidal flux at magnetic axis and plasma boundary and $a = 1 - \psi/\psi_{ma}$. Circular plasma cross section was chosen with aspect ration $A = 3$.

A plasma configuration and profiles for the parameter set $\epsilon_1 = 1,\ \epsilon_2 = 1,\ \alpha_1 = 1,\ \alpha_2 = 1,\ f_0 = 0.272$ are shown in Fig.9. p_0 was chosen to provide $\beta = 15.2\%$ and $\mu = 0.5075$ for the equilibrium.

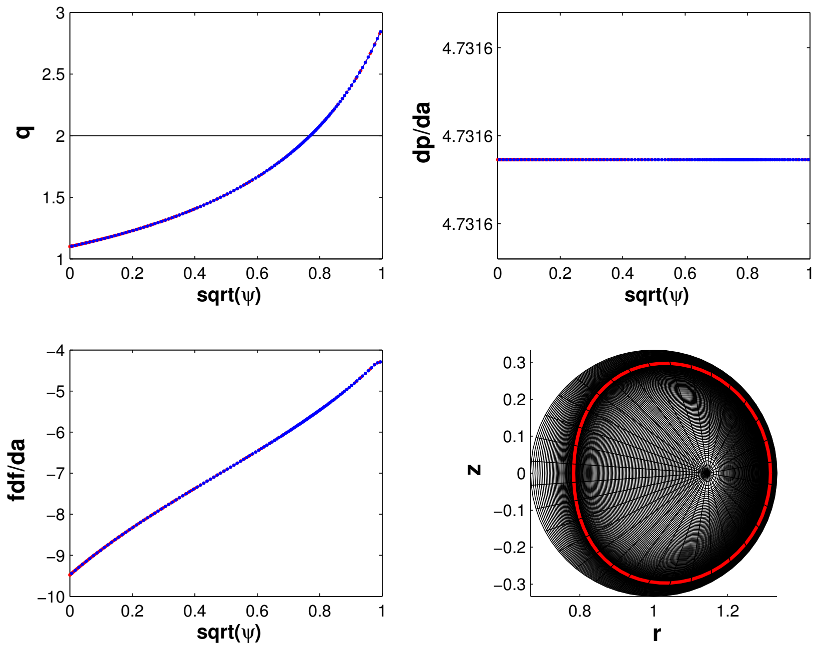


Figure 9: Toroidal equilibrium with profiles (original and fitted by TWIST-R)

Computed Δ' and Γ' demonstrated convergence as shown in Fig.10.

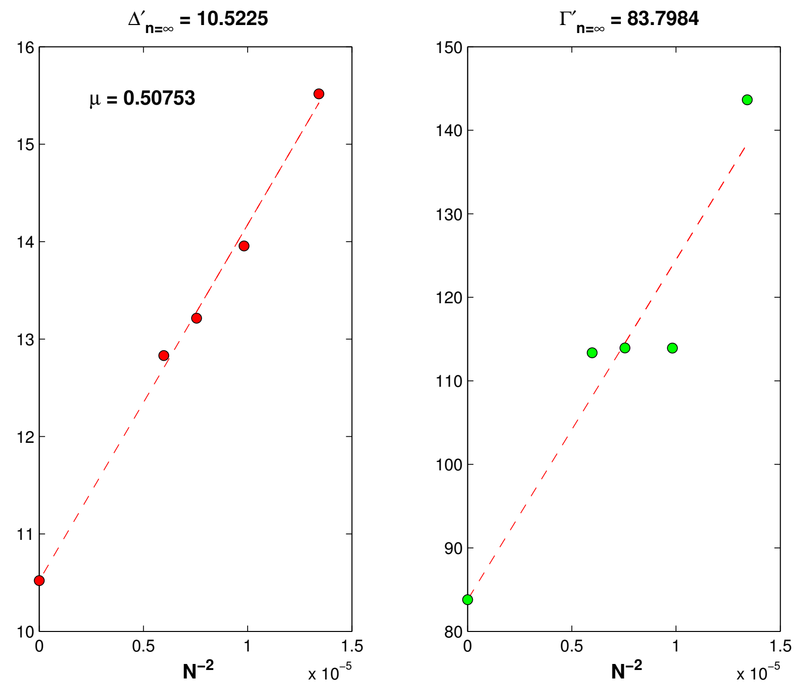


Figure 10: Convergence for toroidal equilibrium with circular cross section

The parameters p_0, ϵ_1, ϵ_2, α_1, α_2 were varied to keep the equilibria ideally stable, what was checked with the ideal MHD stability code TWIST, and a series of finite β equilibria was computed.

Discussion and Summary

The latest version of the TWIST-R code, with many improvements made, provides physically reasonable solutions for the test cases. Investigated Solov'ev equilibria demonstrated near-quadratic convergence in the matching data Δ' and Γ' with radial mesh, over a range of Mercier index $0.5 < \mu < 1.5$ values. The study also showed the expected pole in the tearing mode matching data $\Delta'(\mu)$ at $\mu = 1$. Numerically computed, circular cross section toroidal equilibria were also tested and Δ' and Γ' were computed, with good convergence properties obtained for these cases as well.

2-D IMPLEMENTATION PRODUCES PHYSICALLY REASONABLE SOLUTIONS BUT NUMERICAL DIFFICULTIES REMAIN

- **Solutions appear to be well behaved near resonant surface:**
  - Expected discontinuity in derivative of small Frobenius solution at rational surface is well reproduced
  - Numerically poor behavior of high m harmonics exists near axis:
    - $\Rightarrow$ High m harmonics should vanish like r^m as r -> 0 but vanish much more slowly
    - $\Rightarrow$ Low m harmonics however are well behaved, including resonant m = 2
- **Δ' values are ridiculously large despite the well behaved solutions**
  - Γ' values are apparently reasonable
  - Near quadratic convergence with radial mesh and poloidal harmonics

This is the most pressing remaining issue

- **Solutions are sensitive to the representation used for the equilibrium current density:**
  - Numerical representation of curl(B) or source functions

$\Rightarrow$ Suggests equilibrium accuracy is a crucial issue

This may be source of numerical problem with large Δ' values

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