MOTIVATION

1. Finite orbit effects are of interest in tokamaks when energetic particles are present, and when particles are sufficiently close to the magnetic axis. Fat orbits enhance asymmetry between co- and counter-moving particles and are important in considering transport and rotation drive.

2. It is of interest to have a relatively simple but realistic model in which fat orbits can be dealt with analytically. This would enhance understanding and compliment particle simulation.*

3. Transport near the plasma axis is an unresolved problem and analytic work is needed in this area.

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OUTLINE

(I) Summary of results of present work

(II) Description of model

(III) Analysis of orbit topologies and phase diagram

(IV) Comparison of analytic results with orbit code

(V) Summary and Conclusion

(I) SUMMARY OF RESULTS OF PRESENT WORK

(1) It is demonstrated that the orbit topologies and phase space characteristics of large orbits are radically different from thin orbit considerations with marked asymmetry between co- and counter-moving particles. A concise and complete description of phase space is obtained.

(2) It is shown that in considering large orbit effects, it is advantageous to de-emphasize the poloidal oscillatory motion, but emphasize the radial oscillatory motion because large orbit effects are automatically included from this viewpoint.

(3) Analytic results are obtained for useful time-averaged physical quantities and compare well with orbit code.

(II) DESCRIPTION OF MODEL

- Circular low-β tokamak of constant q_s

$$\vec{B} = \frac{B_0}{h}\left(\theta_p \Theta + \hat{\phi}\right)$$

$$\Psi_T = q_s \psi_p = \frac{B_0 r^2}{2}$$

- Effective vector potential

$$A_{\phi_*} = -\psi_* = -\frac{B_0 r^2}{2q_s} + \frac{v_\parallel}{\Omega} B_0 R_0$$

$$A_{\theta_{p*}} = \frac{B_0 r^2}{2}\left(1 + \frac{2v_\parallel}{\Omega R_0 q_s}\right)$$

$$A_{\psi_{p*}} = 0$$

$$J^{-1} = \nabla\phi \times \nabla\psi_p \cdot \nabla\theta_p$$

$$\dot{\phi} = J^{-1}\frac{v_\parallel}{B_{\parallel *}}\left(\frac{\partial A_{\theta_{p*}}}{\partial \psi_p}\right)$$

$$\dot{\psi}_p = -J^{-1}\frac{v_\parallel}{B_{\parallel *}}\frac{\partial \psi_*}{\partial \theta_p}$$

- **Three constants of motion**

$$E = \frac{v_\parallel^2}{2} + \mu B$$

$$\mu = \frac{v_\perp^2}{2B}$$

$$\psi_* = \psi_p - \frac{I v_\parallel}{\Omega}$$

$$B = \frac{B_0}{h} = \frac{B_0}{1 + \varepsilon \cos\theta_p}$$

- **Three dimensionless parameters**

$$\alpha_p = 1 - \frac{\mu B_0}{E}$$

$$\eta_E = \frac{2E}{\Omega_0^2 R_1^2}$$

$$\eta_\zeta = \frac{\psi_*}{I R_1}$$

$$R_1 = \frac{R_0}{2 q_s}$$

- **Orbits:** $\varepsilon = \dfrac{r}{R_0}$

$$\boxed{\left(\varepsilon^2 - \eta_\zeta\right)^2 = \eta_E \left[\alpha_p + \left(1 + \alpha_p\right) \varepsilon \cos\theta_p\right]} \qquad \text{(I)}$$

(III) ANALYSIS OF ORBIT TOPOLOGIES AND PHASE DIAGRAM

- Orbit topologies are completely determined by solving for Eq. (I) at $\theta_p = 0$ and π

$$\left(\varepsilon^2 - \eta_\zeta\right)^2 = \eta_E\left[\alpha_p \pm \left(1+\alpha_p\right)\varepsilon\right] \qquad \text{(II)}$$

or $$g_\pm = \eta_E\,\varepsilon\left(1+\alpha_p\right) \pm \left[\left(\varepsilon^2 - \eta_\zeta\right)^2 - \eta_E\alpha_p\right] = 0 \qquad \text{(III)}$$

- Four regions of phase space

(I) Four real roots $\{\varepsilon_0, \varepsilon_3, -\varepsilon_2, -\varepsilon_1\}$; $\{-\varepsilon_2, \varepsilon_3\}$ for co-passing particles, $\{\varepsilon_0, -\varepsilon_1\}$ for counter-passing particles

(II) Small energy and pitch angle α_p-trapped particles with two real roots $\{\varepsilon_0, \varepsilon_3\}$ and two complex conjugate roots $\{-\varepsilon_1, -\varepsilon_1^*\}$

(III) and (IV) Co-passing particles only at high energies with two real roots $\{\varepsilon_0, \varepsilon_3\}$ and two complex conjugate roots

Disappearance of trapped-passing boundary implies significant asymmetry between co- and counter-particles

- **Solutions of Eq. (II)**

Define η_ζ by low-field equatorial location of co-passing particle

$$\eta_\zeta \equiv \varepsilon_0^2 - \sqrt{\eta_E[\alpha_p + (1+\alpha_p)\varepsilon_0]}$$

$$\alpha_T \equiv \frac{3}{2\varepsilon_0^2}\sqrt{\eta_E[\alpha_p + (1+\alpha_p)\varepsilon_0]}$$

$$\alpha_E = \frac{27}{2}\frac{\eta_E}{\varepsilon_0^2}$$

(a) In region (I), there are four rots $\{\varepsilon_0, \varepsilon_3, -\varepsilon_1, -\varepsilon_2\}$ given in terms of ε_0 by

$$\varepsilon_3 = \varepsilon_0\left[A\cos\theta_0 - \frac{1}{3}\right]$$

$$\varepsilon_1 = \varepsilon_0\left[\frac{1}{3} - A\cos\left(\theta_0 + \frac{2\pi}{3}\right)\right]$$

$$\varepsilon_2 = \varepsilon_0\left[\frac{1}{3} - A\cos\left(\theta_0 - \frac{2\pi}{3}\right)\right]$$

where $A = \frac{4}{3}\sqrt{1-\alpha_T}$

$$\theta_0 = \frac{1}{3}\tan^{-1}\left(\frac{\sqrt{-D_1}}{C_1}\right)$$

$$D_1 = C_1^2 - (1-\alpha_T)^3$$

$$C_1 = 1 - \frac{3}{2}\alpha_T + \frac{\alpha_E + 6\varepsilon_0\alpha_T^2}{8(1+\varepsilon_0)}$$

(b) In regions (II), (III), and (IV), there are two real roots $\{\varepsilon_0, \varepsilon_3\}$ and two complex conjugate roots $\{\varepsilon_1, \varepsilon_1^*\}$

$$\varepsilon_3 = \varepsilon_0 \left[A_+ + A_- - \frac{1}{3} \right]$$

$$\varepsilon_1 = \varepsilon_0 \left[+ \frac{A_+ + A_-}{2} + \frac{1}{3} + \frac{i\sqrt{3}}{2}\left(A_+ - A_-\right) \right]$$

$$A_\pm = \frac{2}{3}\left[C_1 \pm \sqrt{D_1}\right]^{1/3}$$

- Phase-space boundaries

(a) Trapped-Passing boundary and boundary for potato orbits

$$\alpha_{E_c}^{(\pm)} = \left(1+\varepsilon_0\right)\left[\left(12\alpha_T - \frac{6\varepsilon_0 \alpha_T^2}{1+\varepsilon_0} - 8\right) \pm 8\left(1-\alpha_T\right)^{3/2}\right]$$

$$\alpha_{p_c}^{(\pm)} = \frac{\varepsilon_0}{1+\varepsilon_0}\left[\frac{6\alpha_T^2}{\alpha_{E_c}^{(\pm)}} - 1\right], \qquad \alpha_T < 1$$

(b) Boundary of phase-space at high α_E $\left(\alpha_{p_2}^{(\pm)}\right)$

$$\alpha_{p_2}^{(\pm)} = \sqrt{\frac{2}{(1+\varepsilon_0) \pm \sqrt{(1+\varepsilon_0)^2 - (\alpha_E \varepsilon_0 / 54)}} - 1}$$

(IV) ANALYTIC CALCULATION OF ORBIT AVERAGED QUANTITIES

- **Since radial orbit size is important, consider calculating orbit averaged quantities by emphasizing radial oscillations:**

$$\langle A \rangle = \frac{1}{T} \oint \frac{d\psi_p}{\dot{\psi}_p} A$$

$$T = \oint \frac{d\psi_p}{\dot{\psi}_p}$$

One can show that for $x = \varepsilon^2 \quad x = \varepsilon^2 \left(x_1 = \varepsilon_i^2 \right)$

$$T = \frac{8q_s^2}{\Omega_0} \oint \frac{dx}{\sqrt{g_+ \, g_-}}$$

or

$$\boxed{T = \frac{8q_s^2}{\Omega_0} g_1 K(k)} \qquad \text{(IV)}$$

— **Trapped particles:**

$$g_1 = \frac{1}{\sqrt{A_0 A_3}}$$

$$A_{0,3}^2 = \left(x_{0,3} - x_{1r} \right) + x_{1i}^2$$

$$k^2 = \frac{(x_0 - x_3)^2 - (A_0 - A_3)^2}{4A_0 A_3}$$

— Passing particles: let $y_{ij} = x_i - x_j$,

$$g_1 = \frac{2}{\sqrt{y_{02}\, y_{13}}}$$

$$k^2 = \frac{y_{01}\, y_{23}}{y_{02}\, y_{13}}$$

- Calculation of $\langle \dot{\phi} \rangle$ and $\langle \psi_p \rangle$ radial size is useful

$$\boxed{\begin{aligned} \langle \dot{\phi} \rangle &= \frac{\Omega_0}{4q_s}\left[-\eta_\zeta + \frac{\mathcal{I}_1}{\mathcal{I}_0}\right] \\ \langle \psi_p \rangle &= I\mathcal{R}_1 \frac{\mathcal{I}_1}{\mathcal{I}_0} \end{aligned}} \qquad \text{(V)}$$

— Passing particles ("+" = co, "–" = counter)

$$\frac{\mathcal{I}_\pm}{\mathcal{I}_0} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \pm \sqrt{y_{02}\, y_{13}} \left[\frac{E(k)}{K(k)} F(\xi_\pm, k') + E(\xi_\pm, k') - F(\xi_\pm, k')\right]$$

$$\xi_+ = \sin^{-1}\sqrt{\frac{y_{02}}{y_{03}}}$$

$$\xi_- = \sin^{-1}\sqrt{\frac{y_{13}}{y_{03}}}$$

— Trapped particles

Write $$\varepsilon_3 = \varepsilon_0 (1 - \Delta_R)$$

$$\varepsilon_1 = \varepsilon_0 \left(1 - \frac{\Delta_R}{2} + \frac{i\sqrt{3}}{2} \, \Delta_I \right)$$

$$S_0 = \left(1 - \frac{\Delta_R}{2} \right), \qquad S_1 = \left(1 - \frac{\Delta_R}{4} \right)\left(1 - \frac{3\Delta_R}{24} \right)$$

$$S_2 = \frac{\Delta_R}{S_0^2}, \qquad \Delta_s^2 = \Delta_R^2 + 3\Delta_I^2$$

then

$$\boxed{\begin{aligned} \frac{\mathscr{I}_{1t}}{\mathscr{I}_{0t}} \simeq \varepsilon_0^2 S_0^2 \Bigg\{ & \left(1 - \frac{3\Delta_I^2}{4S_0^2} \right) + \frac{\Delta_s \sqrt{S_1}}{S_0^2} \left[1 - \frac{9\Delta_I^2 S_0^2 (1 - 5S_2 / 48)}{32 S_1^2} \right] \\ & \cdot \left[\frac{E(k)}{K(k)} F(\psi, k') + \{ E(\psi, k') - F(\psi, k') \} \right. \\ & \left. - \frac{1}{K(k)} \tan^{-1}\left(\sqrt{\frac{S_1}{3}} \, \frac{\Delta_R}{\Delta_I S_0} \right) \right] \Bigg\} \end{aligned}}$$

$$\psi \simeq \sin^{-1}\left[\frac{\Delta_s}{4(1 - \Delta_R / 2)} \right]$$

$$\begin{aligned} \lim_{\frac{\eta_E}{\varepsilon_0^3} \ll 1} \langle \dot{\phi} \rangle \simeq \frac{v^2 q_s (1 + \alpha_p)}{\Omega_0 R_0 r_0} & \left[\frac{2E(k)}{K(k)} - 1 \right. \\ & \left. - \frac{1}{K(k)} \sqrt{\frac{32\varepsilon_0^3}{\eta_E (1 + \alpha_p)}} \tan^{-1} \sqrt{\frac{4\eta_E \{ \alpha_p + (1 + \alpha_p) \varepsilon_0}{27 \varepsilon_0^4 D}} \right] \end{aligned}$$

SUMMARY AND CONCLUSIONS

1. In a simple tokamak geometry, we analyzed the orbit topology in the regime where the orbit width is not small compared with the minor radius. A complete phase diagram can be deduced which concisely describes the whole phase space. The results differ markedly from thin orbit theory, and indicate strong asymmetry between the co- and counter-moving particles.

2. When the orbit is fat, one must include the orbit width in doing analytic calculations. a reasonable approach is to emphasize the radial oscillation of the particles rather than the poloidal oscillations. In doing so, we find that the equations of motion reduce to quadratures in the simple model considered. One is then able to solve completely for the time averaged quantities such as the poloidal flux function and the toroidal angular precession. It then appears advantageous to consider transport of ions as transport of radial oscillators.

3. Analytic formulae for time averaged toroidal precession and the poloidal flux function are obtained which agree reasonably well with particle codes. In particular, the energy dependence of the oscillator constant, k, is verified.