Properties of kinetic Reversed Shear Alfvén Eigenmodes

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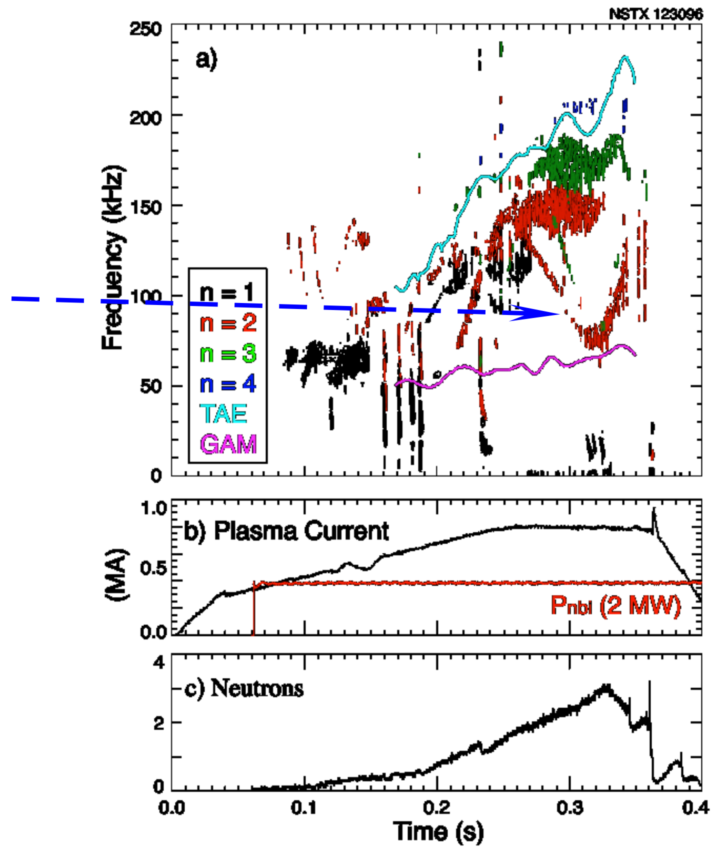
PPPL
PRINCETON PLASMA PHYSICS LABORATORY

Motivation

NSTX

$B(T)$	0.45
$R_0/a(m)$	$0.85/0.66$
β_{pl}/β_{fast} (%)	$\sim 30/15$
$fast\ ions$	$2MW\ NBI$

- Down sweeping RSAEs are rarely observed:
  - is it due to strong damping or
  - due to nonexistence?
- DIII-D and NSTX demonstrate strong losses when RSAEs are present.
- Both bottom and down sweep RSAEs offer a unique opportunity for plasma diagnostics, MHD spectroscopy.



E.D.Fredrickson

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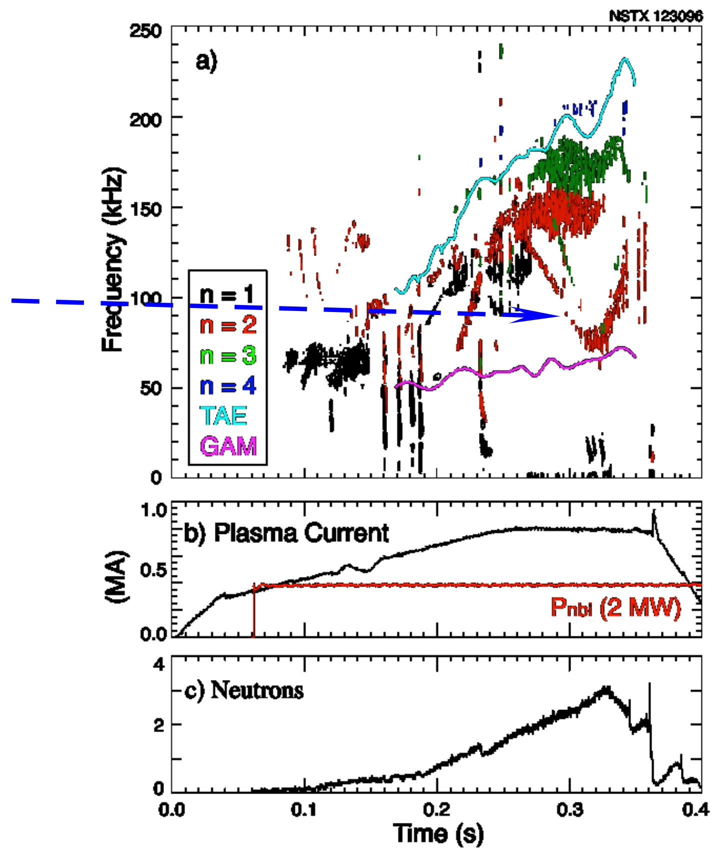

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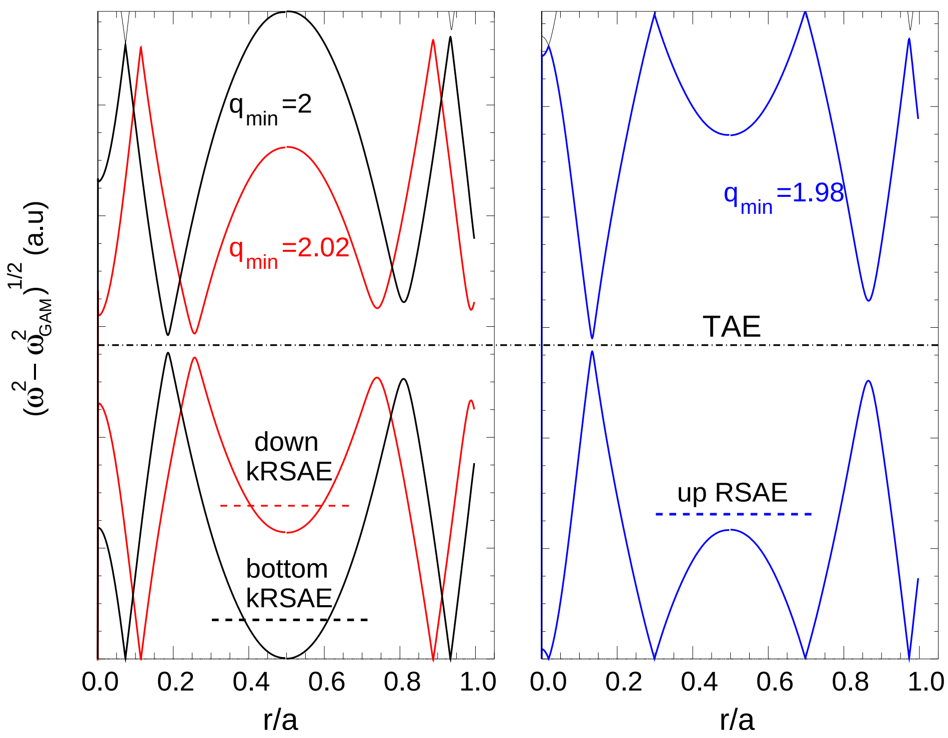
What are the properties of these modes?

Relation of RSAEs and Alfvén continuum during q-profile relaxation



- $n=10$, large aspect ratio cylindrical plasma
- $q = q_{min}/\left[1-(r-r_0)^2/a^2w^2\right]$
- $R/a = 10/1$, $w=2$, $\beta(\%) = 0.1\left(1-r^2/a^2\right)$.
- We will study *down sweeping* and *bottom of the sweep* modes.

Why low-f instabilities are important?

- RSAEs and new class of instabilities, Beta-induced Alfvén Acoustic Eigenmode (BAAE), help to study two fundamental MHD and kinetic waves: Alfvén and acoustic.
- Energetic particle driven low-f MHD instabilities mostly result in radial particle transport:
  - On NSTX, bursting low-f modes can lead to a significant loss of injected beam ions (Fredrickson'06).
- MHD spectroscopy application for plasma diagnostic:
  - RSAE in low- to medium- β plasma
  - BAAE high-β plasma, such as in STs when RSAEs are suppressed.
- Due to coupling to acoustic branch strong interaction with thermal ions is expected:
  - $\Rightarrow$ strong drive due to fast ions and strong damping due to thermal ions,
  - $\Rightarrow$ potential for energy channeling from beam ions directly to thermal ions (α-channeling, Fisch'93, hot-ion mode, Li-Wall).

Commonly used RSAE eigenmode equation in ideal MHD

Basic equation for dominant poloidal harmonic, m.

Coupling to $m \pm 1$ is included in the derivation
(Berk'01, Breizman'03):

$$\partial_r \left[\bar{\omega}^2 - k_0^2(r)\right] \partial_r \phi - m^2 \left[\bar{\omega}^2 - k_0^2(r)\right] \phi + 2m^2 \left(\hat{Q} + \hat{Q}_k\right) \phi = 0.$$

$\hat{Q}_k$ is from kinetics. Eigenmodes exist due to $\hat{Q} > 0$. In MHD:

$$\hat{Q} \simeq 2\alpha \frac{2\bar{\omega}^2 \Delta' - \alpha k_{00}^2}{1 - 4k_{00}^2 q_{min}^2} + \frac{\varepsilon\alpha}{q_{min}^2}\left(1 - \frac{1}{q_{min}^2}\right) + \bar{\omega}^2 \frac{\varepsilon\left(\varepsilon + 2\Delta'\right)}{1 - 4k_{00}^2 q_{min}^2},$$

$k_{00} = k_0(r_0)$, $k_0 = k_\parallel\big|_m = m/q - n$, $q(r_0) = q_{min}$, $\bar{\omega} = \omega R / v_A$.

Bottom RSAE case dominant (2nd) term is due to pressure gradient *(Fu'06, Gorelenkov'06)*

RSAE eigenmode equation in ideal MHD (part 2)

Introduce new variable $z^2 = (r - r_0)^2 m^2/r_0^2$, $\varepsilon > 0$,

$$\frac{\partial}{\partial z}\left(1-z^2\right)\left(1+\mu z^2\right)\frac{\partial}{\partial z}\phi - S\left(1-z^2\right)\left(1+\mu z^2\right)\phi + Q\phi + Q_k\phi = 0$$

where $k_{00} \leq 0$ for down, bottom sweep RSAEs.

- $\mu \simeq 0$ for down sweep RSAEs
- $\mu = 1$ for sweep bottom RSAEs

Down, threshold condition, $\bar{\omega} \simeq k_{00}$:
RSAE is emerging from the continuum: $S \ll Q$

For near the bottom RSAEs:

$$S = \sqrt{\mu/B} = \frac{m q_{min} w^2}{r_0^2}\left(\bar{\omega} + k_{00}\right); \quad Q_{bott} \simeq \frac{n w^2}{\left(\bar{\omega} - k_{00}\right) r_0^2}\alpha\varepsilon\frac{q^2-1}{q^2}.$$

Formal RSAE solution in ideal MHD

Analytically treatable frequency down sweeping case, near threshold $Q \gg S > 0$:

$$\frac{\partial}{\partial z}\left(1-z^2\right)\frac{\partial}{\partial z}\phi - S\phi + Q\phi = 0$$

with zero boundary conditions at $z \to \pm\infty$ formally RSAE solution is

Legendre functions $Q_l(z);\ P_l(z)$

which have expected $\ln|1-z|$ singularity at $z=1$,

and dispersion:

$$Q - S = l(l+1).$$

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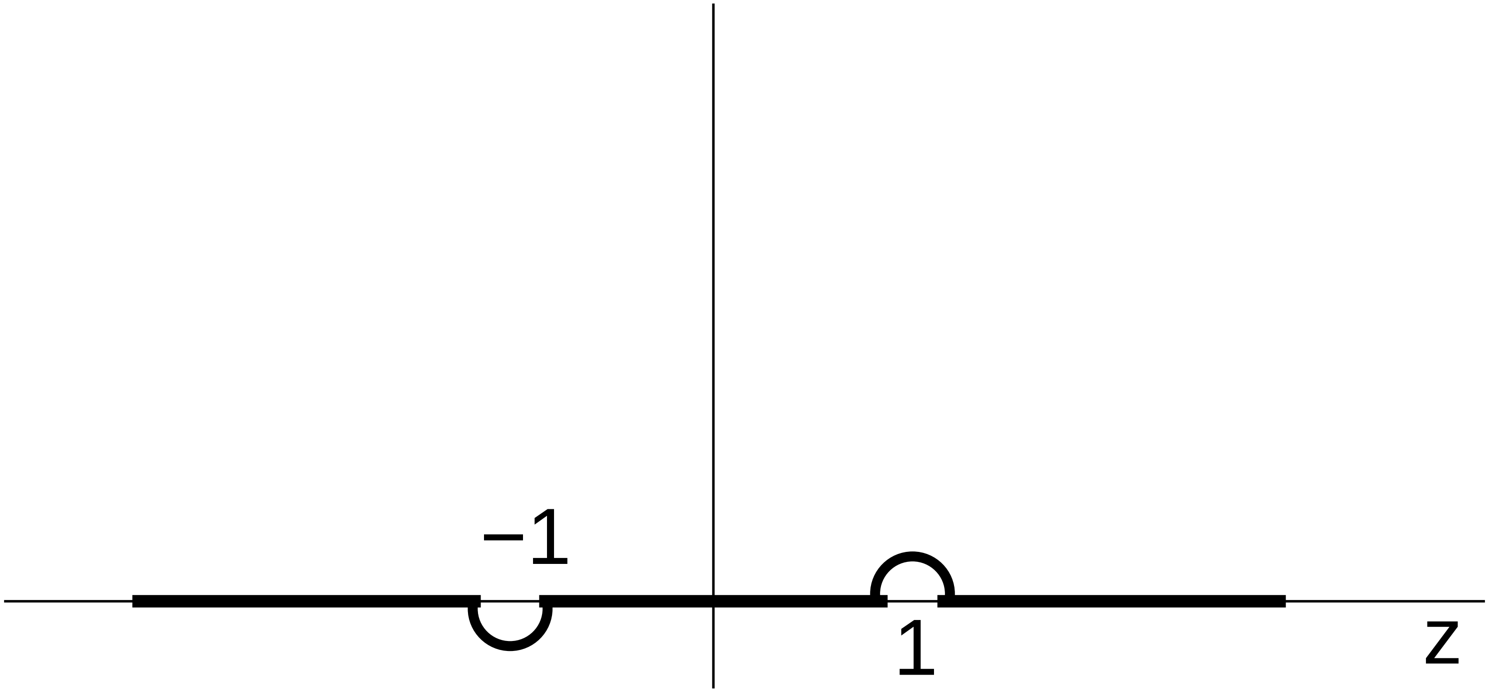
These solutions are not physical! Why?

Standard analytic extension of RSAEs due to causality condition

Account for singularities at $z = \pm(1 + i\varepsilon)$ or positive growth rates:

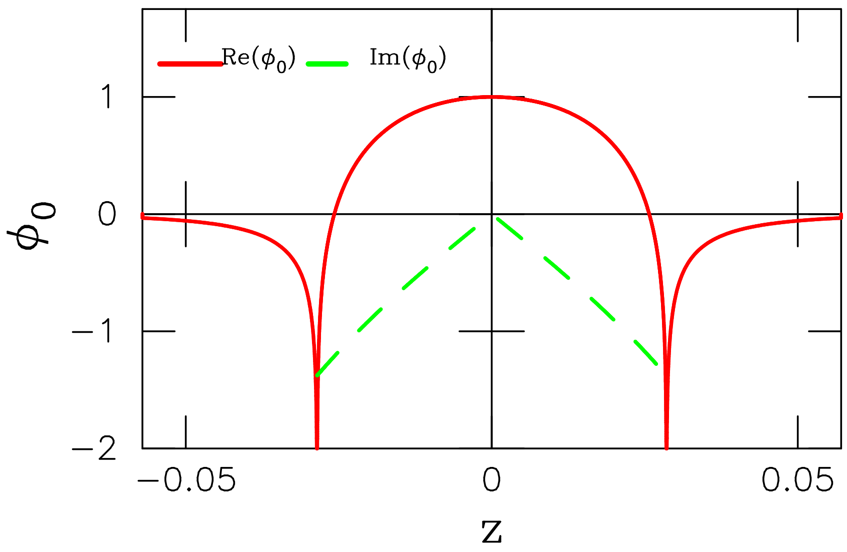


- The contour implies the following transformation rule from $z > 1$ to $z < 1$:

$$\ln(z-1) \rightarrow \ln(1-z) - i\pi.$$

Inconsistency of ideal extended MHD solution

Analytic extension of RSAE solution implies symmetric radial mode structure:



- eigenfrequency is well determined
- real part is symmetric
- $Q_0(z)$ has expected singularities at $z = \pm 1$.
- imaginary part does not match near origin.

Self-consistent theory needs to include kinetic effects *(similar to Timofeev'RoPP75)*.

Kinetic RSAEs (kRSAE) eigenmode equation

Have to include FLR effects $\Rightarrow 4^{th}$ order DE - Orr-Sommerferld equation (hydrodynamic stability of the inhomogeneous flow, flute instbility, *Timofeev NF'68,FyzPl'76*):

$$\lambda^2 \frac{\partial}{\partial z} \lambda^{-2} \frac{\partial^3}{\partial z^3} \phi + \lambda^2 \left(\frac{\partial}{\partial z} D \frac{\partial}{\partial z} - SD + Q \right) \phi = 0,$$

$$D = \left(1 - z^2\right)\left(1 + \mu z^2\right),$$

- Make use of large $\lambda \gg 1$ ($\rho_i \ll a$, $n \gg 1$):

$$\lambda^{-2} = \frac{\rho_i^2}{w^2} \frac{n}{\left(\bar{\omega}^2 - k_{00}^2\right)\left(\bar{\omega} + k_{00}\right)} \left[\frac{3}{4} \bar{\omega}^2 + k_0^2 \frac{T_e}{T_i} \left(1 - i\delta_e\right) \right]$$

(Rosenbluth, Rutherford, PRL'75)

- Use asymptotic solutions of Orr-Sommerfeld equations near $z = \pm 1$.

kRSAE asymptotic behavior around "turning" points $z = \pm 1$

Based on earlier works on Bernstein waves
(Peratt'72, Rabenstein'58, Wasow'48)

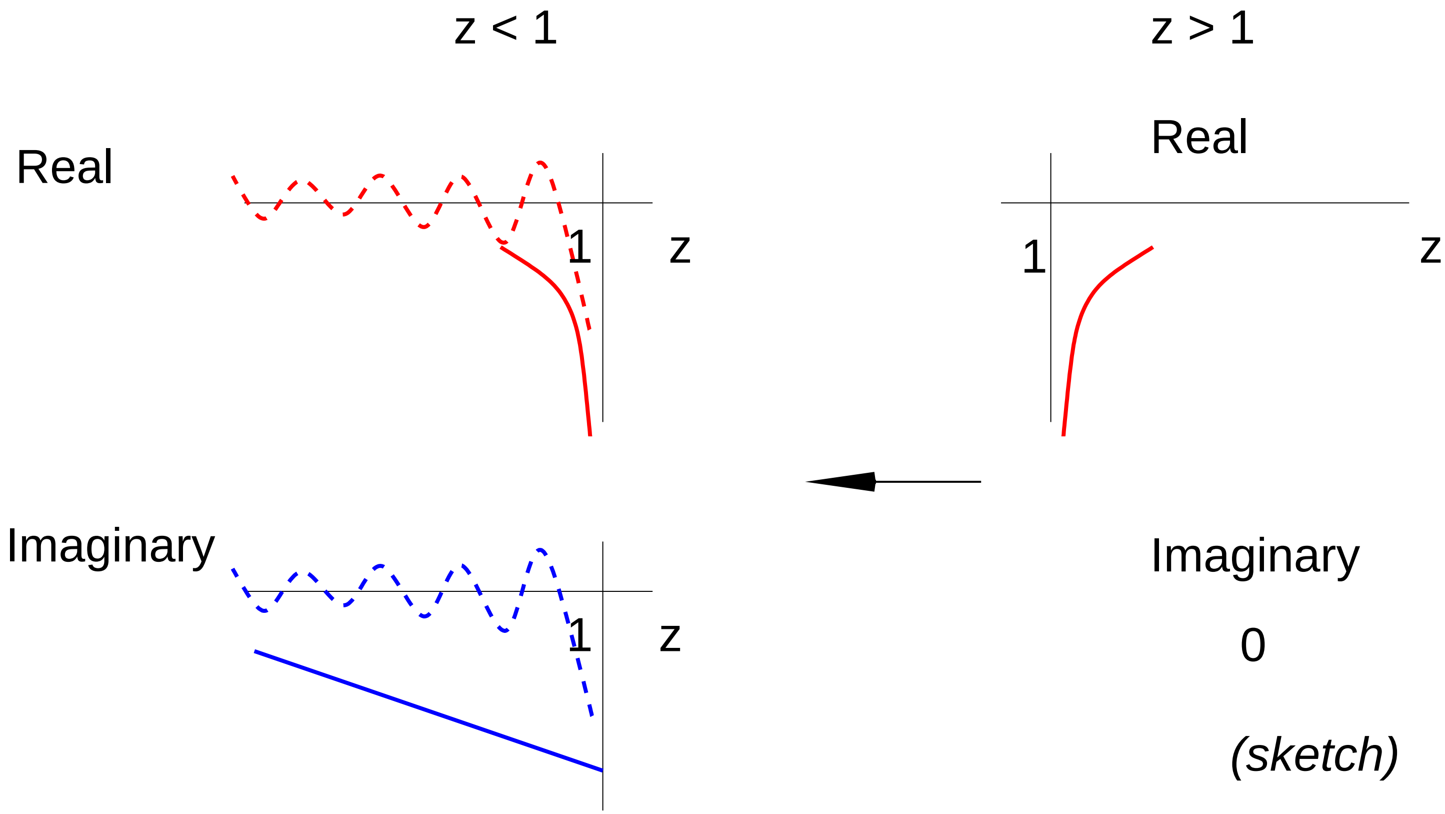
Cross the $z = 1$, $|1 - z| < \lambda^{-2/3}$, point according to

$$\ln(z-1) \rightarrow \ln(1-z) \pm i\pi \pm i\sqrt{\pi}\frac{\exp\left[\mp i\left(5\pi/4 + 2\lambda\sqrt{p}(1-z)^{3/2}/3\right)\right]}{\lambda^{1/2}p^{1/4}(1-z)^{3/4}}$$

Includes

- slow-varying "MHD" solution: $\ln(1-z)$, $i\pi$
- fast-varying KAW oscillatory solution: $(1-z)^{-3/4}$

MHD and KAW oscillatory solutions are coupled at $z = 1$



Ideal MHD dispersion is still required for slow varying MHD solution.

Inner $|z| < 1$ KAW oscillatory WKB solution

Assume WKB anzats

$$\phi = \exp\left[i \int^{z} k(z)\, dz \right],$$

where $k \sim \lambda \gg 1$, $|z-1| > \lambda^{-2/3}$. Solving

$$\lambda^2 \frac{\partial}{\partial z} \lambda^{-2} \frac{\partial^3}{\partial z^3} \phi + \lambda^2 \frac{\partial}{\partial z} D \frac{\partial}{\partial z} \phi = 0,$$

we find for KAW part

$$\phi \sim \frac{\sqrt{\pi p}\lambda_1}{\lambda^{3/2} D^{3/4}} \exp\left[-i\frac{3\pi}{4} - i \int_z^1 \lambda \sqrt{D} dz \right]$$

which has the same asymptotic as the oscillatory solution at $z = 1$.

Matching oscillatory left/right solutions near $z = 0$ results in kRSAE dispersion (imaginary part)

Oscillatory part has to decrease (dissipate) to match the MHD imaginary part at $z = 0 \Rightarrow$ *collisonless damping*:

$$\Im \int_0^1 \lambda \sqrt{D} dz \simeq \frac{1}{2} \ln \left(\frac{\lambda_0 \pi}{\lambda_1^2 p} \right) < 0.$$

Dissipation implies damping rate (assuming $|\bar{\gamma}| \ll \bar{\omega} + k_{00}$):

$$\bar{\gamma} \simeq \frac{\rho_i \sqrt{n}}{w} \left[3 + \frac{4T_e}{T_i} \frac{k_{00}^2}{\bar{\omega}^2} \right]^{1/2} \frac{\bar{\omega}^2}{(\bar{\omega} - k_{00})^{3/2}} \ln \left(\frac{\lambda_0 \pi}{\lambda_1^2 p} \right) \frac{1}{\int_0^1 \sqrt{D} dz} < 0.$$

$w^2 = 2q / \left. q'' \right|_{q=q_{min}}$.

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Ideal MHD limit is weakly damped!

$$\bar{\gamma} = O(\rho_{i*} \ln \rho_{i*}) \to 0$$

"MHD" singular layer damping model is not correct for kRSAEs.
Dissipation is within $\Delta z = \ln \rho_{i*}$ - non perturbative.

Several kRSAEs can be present

Real part matching condition splits the mode into several eigenmodes

$$\Re \int_0^1 \lambda \sqrt{D} dz \simeq \frac{\pi}{4} + 2l\pi,$$

$l \gg 1$ is integer.

kRSAEs split in sub frequencise within $-\bar{\gamma} < \bar{\omega} - \bar{\omega}_{MHD} < \bar{\gamma}$ if $|\bar{\omega} - k_{00}| > \bar{\gamma}$

$$\Delta\bar{\omega} \simeq 4\pi \frac{\rho_i \sqrt{n}}{w} \left[3 + \frac{4T_e}{T_i} \frac{k_{00}^2}{\bar{\omega}^2} \right]^{1/2} \frac{\bar{\omega}^2}{(\bar{\omega} - k_{00})^{3/2}} \frac{1}{\int_0^1 \sqrt{D} dz} \ll \bar{\omega}_{MHD}$$

For damped mode with $\bar{\gamma}$ we expect several RSAEs to exist near the MHD frequency $\bar{\omega}_{MHD}$

$$N_{kRSAE} \simeq \frac{-\bar{\gamma}}{\Delta\bar{\omega}} \simeq \frac{-1}{4\pi} \ln \left(\frac{\lambda_0 \pi}{\lambda_1^2 p} \right) \simeq -O\left(\ln \rho_i^2 n / w^2 \right) / 4\pi.$$

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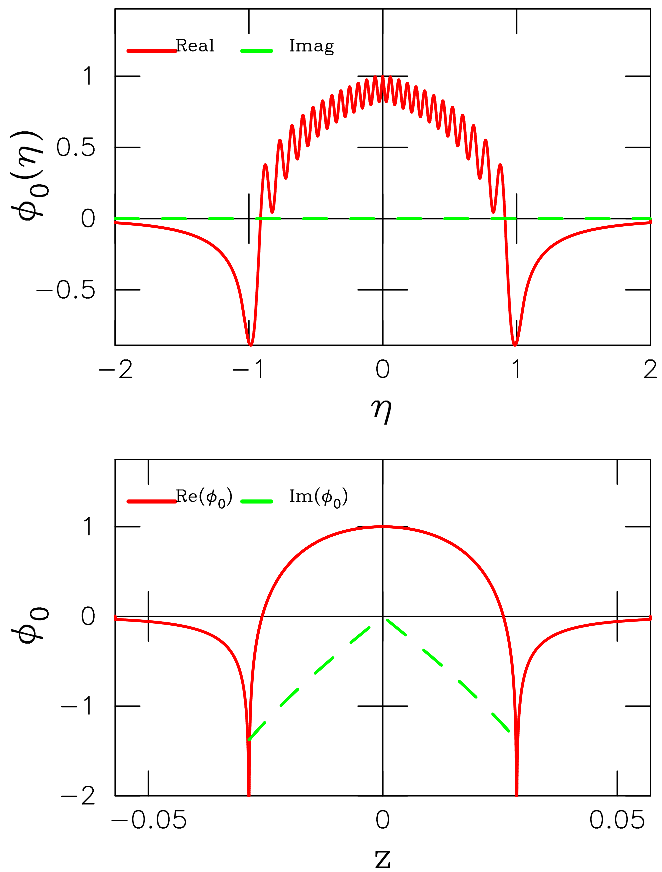
For unstable kRSAE, $\bar{\gamma} = \bar{\gamma}_{drive}$, the number of unstable mode will serve as a way to estimate the growth rate.

Direct numerical solution of kRSAE eigenmode equation vs MHD solution (example)



- Large aspect ratio, low β plasma: $w = 2$, $R/a = 10/1$, $r/a = 0.5$, $m/n = 20/10$, $\beta = 10^{-3}\left(1 - r^2/a^2\right)$, $\lambda = 10^2$.
- Only real part of kRSAE is calculated
  - the same MHD frequency was assumed from MHD solution.

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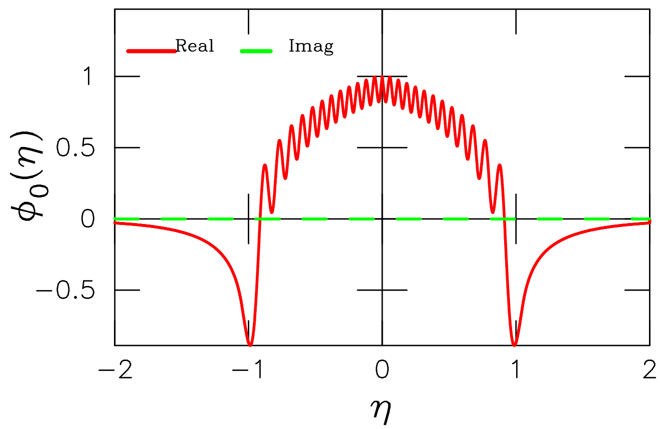

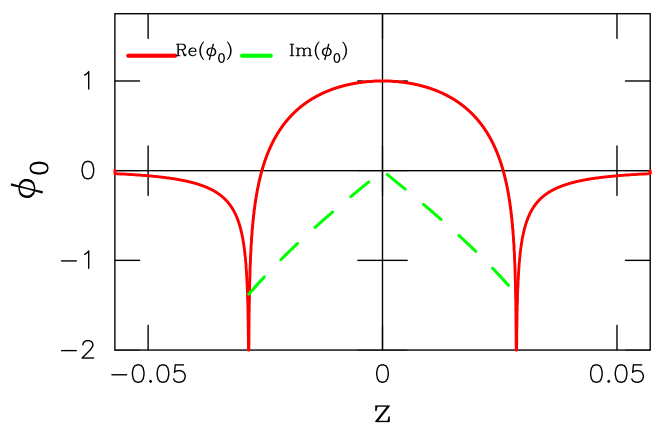

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- Only real part of kRSAE is calculated
  - the same MHD frequency was assumed from MHD solution.

- MHD dispersion can be used for kRSAE frequency with $\pm|\overline{\gamma}|$ accuracy.

MHD dispersion determines kRSAE eigenfrequency

Find only slow varying, MHD, solution:

$$\frac{\partial}{\partial z}\left(1-z^2\right)\left(1+\mu z^2\right)\frac{\partial}{\partial z}\phi - S\left(1-z^2\right)\left(1+\mu z^2\right)\phi + Q\phi = 0$$

General (non threshold) eigenmode equation is solved by minimizing the quadratic form with the proper trial function ($\mu = 0$ as example):

$$\phi = \frac{c_0}{2}\left(z\ln\left|\frac{z+1}{z-1}\right| - 2\right) + c_0\frac{i\pi|z|}{2}H(1-|z|)$$

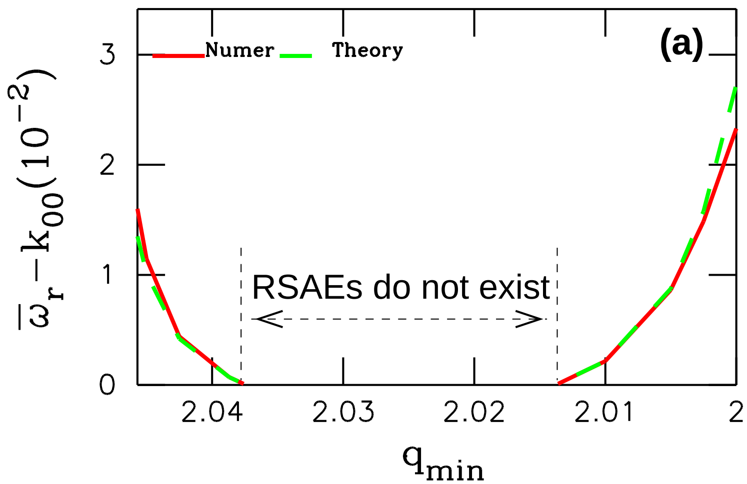
produces following dispersion:

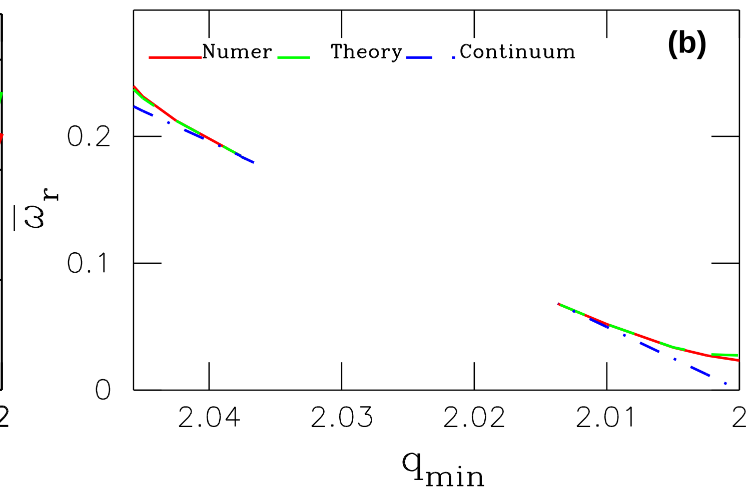
$$\Re\left(Q + \langle Q_k\rangle\right) - 0.4S = 2.$$

Numerical MHD solution frequency is in good agreement with theory solution

q -profile is relaxing from 2.05 to 2.

Theory curve is $Q - 0.4S = 2$.



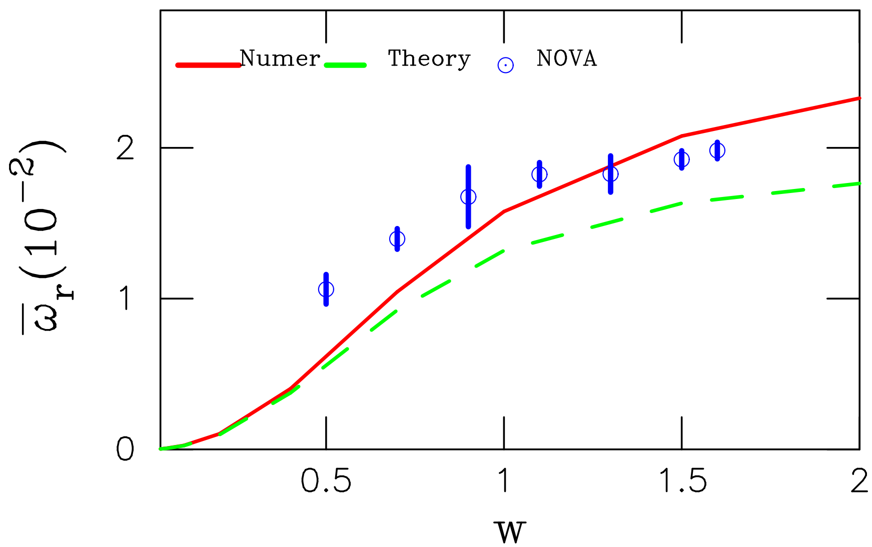


Used large aspect ratio, low beta plasma, $w = 2$, $R/a = 10/1$, $r/a = 0.5$, $m/n = 20/10$, $\beta = 10^{-3}\left(1 - r^2/a^2\right)$.

MHD allows to study some basic properties of kRSAEs:

- existence criteria,
- frequency upshift from the Alfvén/acoustic continuum.

How good big ideal MHD codes are?



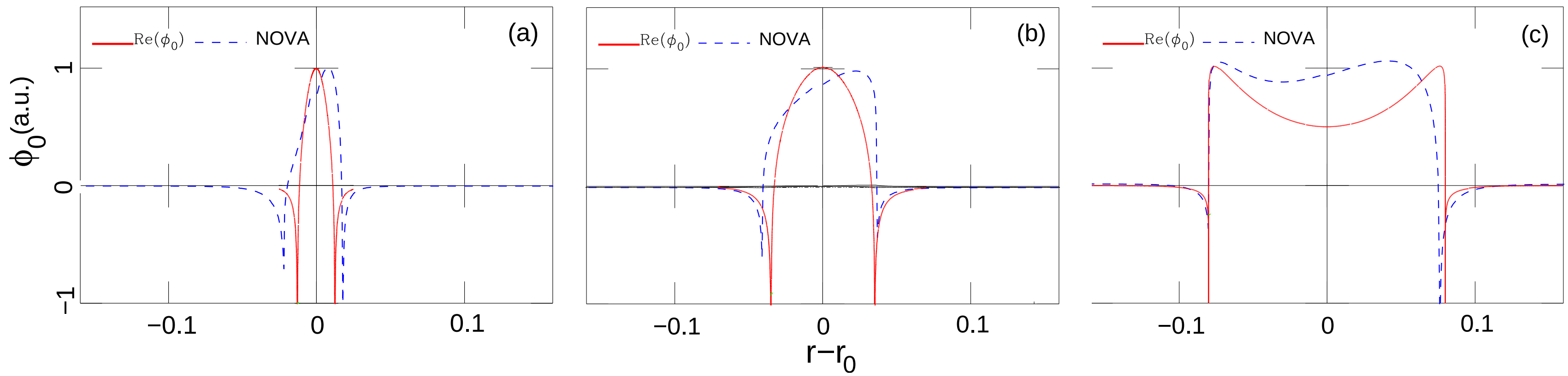
Apply NOVA for the same plasma conditions

- finite element code,
- apply for sweep bottom kRSAE,
- vary the q-profile flatness: higher w more flat q-profile is, $w^2 = 2q/ q''|_{q=q_{min}}$
- compare with the direct shooting simulations: Numer. curve.

Theory (WKB, MHD) frequency $\bar{\omega} = -\frac{\pi^2}{2^3}\frac{r_0^2}{mqw^2} + \sqrt{\frac{\pi^4}{2^6}\frac{r_0^4}{m^2q^2w^4} + \frac{\alpha\varepsilon}{q^2}\left(1 - \frac{1}{q^2}\right)}$

has limit of quasi-local approximation if $m \gg 1$, or $w \gg r_0$ (neglecting 2^{nd} derivative r-dependence, Breizman,Varenna'06).

NOVA produces RSAE structure similar to shooting code results



- NOVA gives approximate kRSAE frequency in agreement with theory.
- Despite having finite elements it seems to pick correct slow verying asymptotics near $z = \pm 1$.
- Radial localization of the mode is predicted.
- Can be used for perturbative addition of the kinetic effects.

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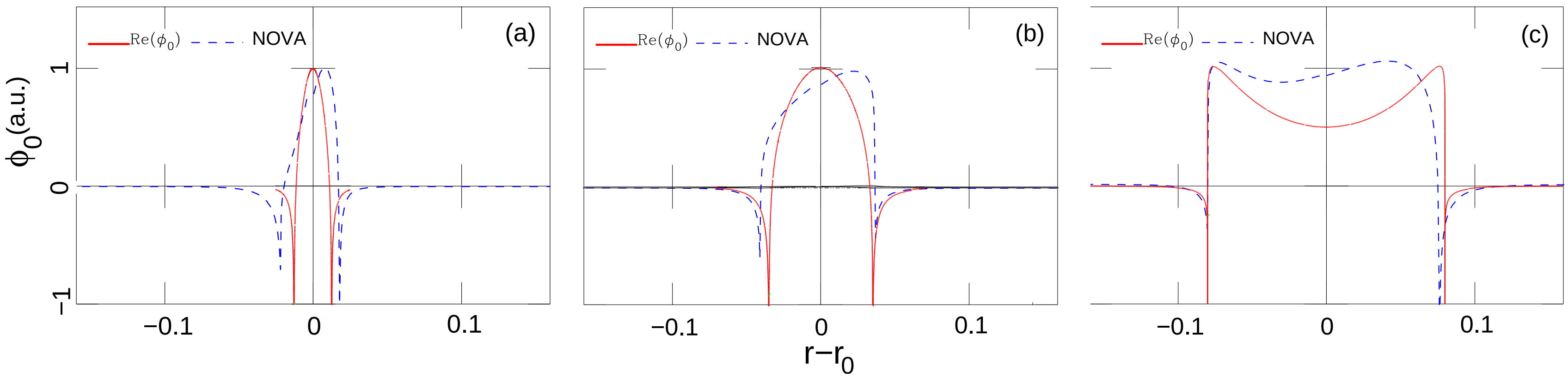

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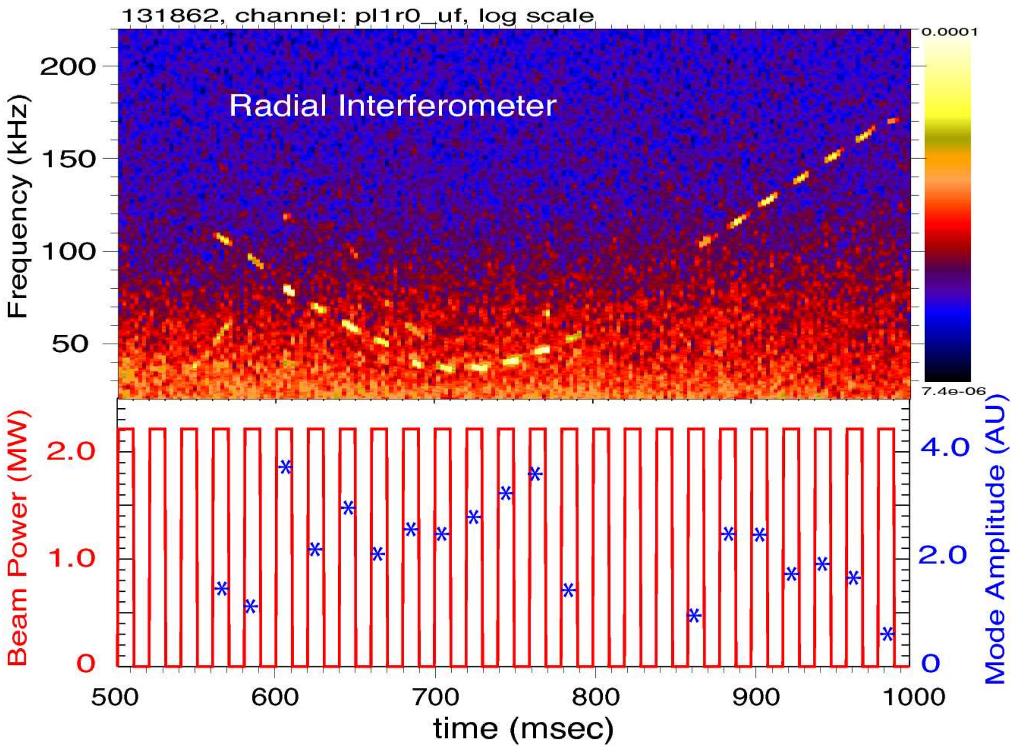
To properly treat continuum resonances ideal MHD codes (NOVA) need kinetic extension

DIII-D experimental evidence of weakly damped bottom/down kRSAEs (PPPL/DIII-D colaboration, R.Nazikian)



DIII-D plasma at $q_{min} = 2$, $n = 1$.

- Weak time dependence of the amplitude of RSAE from down-bottom-up, and
- instantaneous response to $10 msec$ NBI blips

$\Rightarrow$ may suggest weak damping.

This theory selects frequency of down kRSAE and gives $\bar{\gamma} \sim -\frac{\rho_i \sqrt{n}}{w}\sqrt{\bar{\omega}}$.

Summary

- Theory is developed for sweeping down and bottom of the sweep kRSAE modes.
  - kinetic eigenmodes are shown to exist
  - their frequency ($\sim \omega_{MHD}$) and continuum/radiative damping rate are derived
  - kRSAEs are not damped in the zero FLR limit
  - each kRSAE can be splitted into sub-modes within $\gamma < \omega - \omega_{MHD} < \gamma$ range,
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  - existence criteria of down sweeping RSAEs is developed.
- kRSAEs can extend MHD spectroscopy
  - bottom of the sweep kRSAE frequency gives information about the pressure gradient, acoustic mode effects,
  - theory helps to separate finite pressure and pressure gradient effects,
  - frequency splitting should be seen, contains information about the drive,
  - multiple kRSAEs with close frequency should enhance fast ion transport.