A MODEL OF ELECTRON TRANSPORT FROM SELF-CONSISTENT ACTION-ANGLE TRANSPORT THEORY [1]

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Main references

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MOTIVATION AND GOAL

Motivations

- *Main motivation* is to explore the predictions of the ***self-consistent action-angle transport theory*** on axisymmetric transport
  - We do this by addressing two concrete issues in electron transport not yet fully understood: (1) ***peaked density profile with no central sources*** (pinches); (2) ***current density transport***

Goals

- Set up ***a procedure*** which goes from the initial collision operator in action-space to an *explicit set* of transport equations which could be tested against experimental data
- Study ***electron*** transport in ***magnetic turbulence*** keeping the "drive" from the ***safety factor gradient*** together with the more conventional thermodynamic drives

ACTION-ANGLE APPROACH TO AXISYMMETRIC TRANSPORT

Use of actions and normal modes

Collision operators in action-space have been introduced to obtain a ***very general*** and at the same time ***formally simple*** description of transport phenomena in ***complex geometries*** due to ***electromagnetic spectra***.

This has been achieved by:

(1) using ***action-angle variables*** to describe particle motion, so to automatically include orbit and mode complexities pertinent to *inhomogeneous geometries*

(2) expressing the fields as an ***expansion over plasma normal modes***, so to permit a *formal solution of Maxwell's equations*

Actions

Canonical transformation from $(\mathbf{x},\mathbf{p})$ to $(\mathbf{J},\boldsymbol{\Theta})$ where the actions $J_i \equiv \oint dx_i\, p_i$ are constant of the unperturbed motion, and $\boldsymbol{\Theta}$ are cyclic:

$$H(\mathbf{J},\boldsymbol{\Theta};t) = H_0(\mathbf{J};t) + h(\mathbf{J},\boldsymbol{\Theta};t) + \cdots$$

- Unperturbed motion (assumed *integrable*) and perturbed motion

$$H_0(\mathbf{J};t) \Rightarrow \begin{cases} \dot{\mathbf{J}}_0 = -\partial H_0/\partial\boldsymbol{\Theta} = 0 \\ \dot{\boldsymbol{\Theta}} = \partial H_0/\partial\mathbf{J} \equiv \boldsymbol{\Omega}(\mathbf{J};t) \end{cases} \qquad h(\mathbf{J},\boldsymbol{\Theta};t) \Rightarrow \begin{cases} \delta\dot{\mathbf{J}} = -\partial h/\partial\boldsymbol{\Theta} \\ \delta\dot{\boldsymbol{\Theta}} = \partial h/\partial\mathbf{J} \approx 0 \end{cases}$$

- **Magnetized plasma slab:** $\mathbf{B} = B(x)\hat{\mathbf{z}} = \partial_x A_y(x)\hat{\mathbf{z}}$:

$$\mathbf{J} \rightarrow (J_g, p_y, p_z) \qquad \boldsymbol{\Theta} \rightarrow (\Theta_g, Y, z)$$

where (X,Y)= G.C. coordinates, and

$$\begin{aligned} J_g &= \frac{Mc}{e}\mu_0 \quad \text{gyro} - \text{action} \\ p_y &= Mv_y + \frac{e}{c}A_y(x) \equiv \frac{e}{c}A_y(X) \quad \text{canonical y} - \text{momentum} \\ p_z &= Mv_z \end{aligned}$$

- ***Axisymmetric toroidal geometry:***

Flux coordinates: $\xi = (\alpha, \theta, \zeta)$ where α is the toroidal flux function $\alpha \equiv \psi_t = \Psi_t/(2\pi)$; θ is the poloidal angle, ζ is the toroidal angle.

Convenient actions are [Ref: A.N. Kaufman, (1972)]:

$$
\begin{aligned}
J_g &= \mu_0 \frac{M^2 c}{q} \quad \text{gyroaction} \\
J_\zeta &= M v_\zeta + \frac{e}{c} A_\zeta(r) \equiv \frac{e}{c} A_\zeta(r_b) \quad \text{canonical toroidal angular momentum} \\
J_b &= \oint \frac{d\theta}{2\pi} \frac{q}{c} \tilde{\alpha}(\theta; H_0, J_g, J_\zeta) \quad \text{toroidal flux inside drift orbit}
\end{aligned}
$$

where $\tilde{\alpha}$ is the projection on the poloidal cross-section of the guiding center trajectory

- *Two J's are* $\mathbf{v}$*-like* $(\overline{v}_\perp, \overline{v}_\parallel)$, and one is $\mathbf{r}$-like (r_b)

  For untrapped particles:

  – J_g is $v_\perp$-like

  – J_ζ is $v_\parallel$-like (particle transit speed)

  – $J_b \approx (e/c)\Psi_t(r_b)$ is r-like (magnetic surface on which particle moves)

- The ***first order Hamiltonian*** is expanded in a Fourier series in the ignorable and periodic coordinates $\boldsymbol{\Theta}$:

$$h = [q\Phi_1 - (q/c)\mathbf{v} \cdot \mathbf{A}_1] \rightarrow \sum_{\boldsymbol{\ell}} h(\boldsymbol{\ell}, \mathbf{J}; t) \exp(+i\boldsymbol{\ell} \cdot \boldsymbol{\Theta})$$

  – The ***triplet of integers*** $\boldsymbol{\ell} = (\ell_g, \ell_b, \ell_\zeta)$ singles out each one of the harmonics of the particle perturbing Hamiltonian, i.e., of the *orbital motion*. In the general case $\boldsymbol{\ell} \neq \mathbf{k}$ (coincidence in zero-gyroradius, driftless limit).

Normal modes

- In the gauge $\Phi = 0$, the two "$\boldsymbol{\nabla}\times$ " Maxwell equations can be cast in the form **(Ref: Kaufman, 1972)**

$$\boldsymbol{\Delta}(\mathbf{x},\omega)\cdot\mathbf{A}_1(\mathbf{x},\omega) = -\left(\frac{c}{\omega}\right)^2 \frac{4\pi}{c}\mathbf{j}_{\mathrm{ext}}(\mathbf{x},\omega)$$

- The inversion is achieved by Green's function method ($\mathbf{G} = \boldsymbol{\Delta}^{-1}$)

$$\mathbf{A}_1(\mathbf{x}) = -\left(\frac{c}{\omega}\right)^2 \frac{4\pi}{c}\int d\mathbf{x}'\ \mathbf{G}(\mathbf{x},\mathbf{x}')\cdot\mathbf{j}_{\mathrm{ext}}(\mathbf{x}')$$

  where $\boldsymbol{\Delta}(\mathbf{x},\omega)\cdot\mathbf{G}(\mathbf{x},\mathbf{x}',\omega) = \mathbf{I}\delta(\mathbf{x}-\mathbf{x}')$

- This has been solved in 3-D **(Ref: Mynick, 1988)** by representing $\mathbf{E}_1 = (i\omega/c)\mathbf{A}_1(\mathbf{x},\omega)$ in terms of the basis set $\mathbf{E}_a(\mathbf{x})$ of normal modes (i.e., $\mathbf{j}_{\mathrm{ext}} = 0$), so that $\mathbf{G}$ and $\boldsymbol{\Delta}$ are brought into 'diagonal form', making the inversion of $\boldsymbol{\Delta}$ simple

$$\mathbf{G}(\mathbf{x},\mathbf{x}',\omega) = \sum_a \frac{\mathbf{E}_a(\mathbf{x})\mathbf{E}_a(\mathbf{x}')}{N_a\Delta_a(\omega)}$$

  where $G_a(\omega) = \Delta_a^{-1}(\omega)$ are the eigenvalues of the eigenvalue problem for Maxwell operator: $\mathbf{G}\cdot\mathbf{E}_a(\mathbf{x}) = G_a(\omega)\mathbf{E}_a(\mathbf{x})$

QUASI-LINEAR AND SELF-CONSISTENT COLLISION OPERATOR

Quasi-linear collision operator

Evolution of the 0^{th} order part of $\boldsymbol{\Theta}$-averaged distribution function [(Ref: Kaufman, 1972)]:

$$\frac{\partial f(\mathbf{J};t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \boldsymbol{\cdot} \langle \delta \dot{\mathbf{J}} \delta f \rangle_{\Theta} = \frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{D}^{ql}(\mathbf{J};t) \cdot \frac{\partial f(\mathbf{J};t)}{\partial \mathbf{J}} \; .$$

The ***diffusion tensor*** is

$$\begin{aligned} \mathbf{D}^{ql}(\mathbf{J}) &= \int_0^{\infty} d\tau \; \langle \dot{\mathbf{J}}[\mathbf{J}(t), \boldsymbol{\Theta}(t), t] \; \dot{\mathbf{J}}[\mathbf{J}(t-\tau), \boldsymbol{\Theta}(t-\tau), t-\tau] \rangle_{\Theta} \\ &= \sum_{\boldsymbol{\ell}} \boldsymbol{\ell}\boldsymbol{\ell} \sum_a 2\pi \delta(\boldsymbol{\ell} \boldsymbol{\cdot} \boldsymbol{\Omega} - \omega_a) |C_a^{ql}(\boldsymbol{\ell}, \mathbf{J}; \omega_a)|^2 \; , \end{aligned}$$

where the “coupling coefficient” $C_a^{ql}(\boldsymbol{\ell}, \mathbf{J}, \omega_a) \equiv h_a(\boldsymbol{\ell}, \mathbf{J}, \omega_a)$ describes the ***energy exchange between a perturbing wave and the unperturbed particle motion***

Self-consistent (or generalized Balescu-Lenard) collision operator

Evolution of the 0^{th} order part of $\boldsymbol{\Theta}$-averaged distribution function (Ref: Mynick, 1988):

$$\frac{\partial f(\mathbf{J}_1;t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[\mathbf{D}(\mathbf{J}_1) \cdot \frac{\partial f(\mathbf{J}_1;t)}{\partial \mathbf{J}_1} - \mathbf{F}(\mathbf{J}_1) f(\mathbf{J}_1;t) \right]$$

where $\mathbf{D}(\mathbf{J};t)$ is the usual (ensemble averaged) ***diffusion tensor***, and

$$\mathbf{F}(\mathbf{J},t) \equiv \langle \dot{\mathbf{J}}^{pol}[\mathbf{J}(t), \boldsymbol{\Theta}(t), t] \rangle_{\Theta}$$

is the ***friction vector*** which considers the polarization field induced by the test particle $\Rightarrow$ *back-reaction of particle on fields*

Diffusion and drag coefficients

The coefficients D and F have been evaluated by **Mynick, 1988** and **Hitchcock, Hazeltine, Mahajan, 1983**:

$$\mathbf{D}(\mathbf{J}_1) = \sum_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} \boldsymbol{\ell}_1 \, \boldsymbol{\ell}_1 \, D_0(\mathbf{J}_1, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) \ , \qquad \mathbf{F}(\mathbf{J}_1) = \sum_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} \boldsymbol{\ell}_1 \, \boldsymbol{\ell}_2 \cdot \mathbf{F}_0(\mathbf{J}_1, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) \ ,$$

$$D_0(\mathbf{J}_1; \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) = \sum_2 \left(\frac{2\pi}{M_2} \right)^3 \int d\mathbf{J}_2 \ Q(\boldsymbol{\ell}_1, \mathbf{J}_1; \boldsymbol{\ell}_2, \mathbf{J}_2) \ f(\mathbf{J}_2)$$

$$\mathbf{F}_0(\mathbf{J}_1; \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) = \sum_2 \left(\frac{2\pi}{M_2} \right)^3 \int d\mathbf{J}_2 \ Q(\boldsymbol{\ell}_1, \mathbf{J}_1; \boldsymbol{\ell}_2, \mathbf{J}_2) \ \frac{\partial f(\mathbf{J}_2)}{\partial \mathbf{J}_2} \ ,$$

$$Q(\boldsymbol{\ell}_1, \mathbf{J}_1, \boldsymbol{\ell}_2, \mathbf{J}_2) \equiv 2\pi \ \delta(\boldsymbol{\ell}_1 \cdot \boldsymbol{\Omega}_1 - \omega) \ \sum_a |C_a(\boldsymbol{\ell}_1, \mathbf{J}_1, \boldsymbol{\ell}_2, \mathbf{J}_2, \omega)|^2 \Big|_{\omega = \boldsymbol{\ell}_2 \cdot \boldsymbol{\Omega}_2}$$

$$C_a(\boldsymbol{\ell}_1, \mathbf{J}_1, \boldsymbol{\ell}_2, \mathbf{J}_2, \omega) \equiv \frac{4\pi \ h_a(\boldsymbol{\ell}_1, \mathbf{J}_1, \omega) \ h_a^*(\boldsymbol{\ell}_2, \mathbf{J}_2, \omega)}{N_a \Delta_a(\omega)} \ .$$

Here, $\Delta_a(\omega)$ is the **eigenvalue of mode "a" of the Maxwell operator** (generalization to inhomogeneous, electromagnetically interacting plasmas of the dielectric function), and N_a = a normalization factor.

TRANSPORT EQUATIONS

- Let $\chi(\mathbf{x},\mathbf{v};t)$ be some quantity of interest, whose ***mean*** w.r.t. the distribution function is

$$\overline{\chi}(\mathbf{x};t) \equiv \int d\mathbf{v}\ \chi(\mathbf{x},\mathbf{v};t)\ f(\mathbf{x},\mathbf{v};t)\ ,$$

and its **flux surface average** is

$$\langle\overline{\chi}\rangle_{\bar{\alpha}}(\bar{\alpha},t) \equiv \frac{1}{\mathcal{V}'(\bar{r})}\int d\mathbf{x}\ \delta[\alpha(\mathbf{x},t)-\bar{\alpha}]\ \overline{\chi}(\mathbf{x};t)\ .$$

with $\bar{\alpha}$ (or $\bar{r}$) the flux surface of interest.

Taking the time derivative and passing to action-angle variables

$$\frac{\partial\langle\overline{\chi}\rangle_{\bar{\alpha}}}{\partial t} = \frac{1}{\mathcal{V}'(\bar{r})\ M^3}\iint d\boldsymbol{J}\,d\boldsymbol{\Theta}\ \underbrace{\frac{\partial}{\partial t}\{\chi(\boldsymbol{J},\boldsymbol{\Theta},t)\ \delta[\alpha(\boldsymbol{J},\boldsymbol{\Theta},t)-\bar{\alpha}]\ f(\boldsymbol{J},\boldsymbol{\Theta},t)\}}_{\text{obtain 3 terms}}$$

- Assume as equilibrium distribution a drifting Maxwellian

$$f(\boldsymbol{J}) = f_M(\boldsymbol{J}) \left\{ 1 + \frac{V_\parallel[\alpha(\boldsymbol{J})]}{T[\alpha(\boldsymbol{J})]} \, P[\alpha(\mathbf{J})] \right\}$$

where $V_\parallel$ is the parallel drift (flow) speed, $P \equiv M v_\parallel$, and

$$f_M(\boldsymbol{J}) = \frac{N[\alpha(\boldsymbol{J})] \; M^{3/2}}{\pi^{3/2} \; 2^{3/2} \; T[\alpha(\boldsymbol{J})]^{3/2}} \; \exp\left(-\{H_0(\mathbf{J}) - q\Phi_0[\alpha(\boldsymbol{J})]\}/T[\alpha(\boldsymbol{J})]\right)$$

- ***Radial transport law*** for χ

$$\overbrace{\frac{\partial}{\partial t}\langle\overline{\chi}_1\rangle_{\bar{\alpha}} - \left\langle\overline{\frac{\partial \chi_1}{\partial t}}\right\rangle_{\bar{\alpha}}}^{time\ evolution} = -\overbrace{\frac{1}{\bar{\mathcal{V}}'}\frac{\partial}{\partial\bar{\alpha}}\bar{\mathcal{V}}'\left\langle\overline{\chi_1\frac{\partial \alpha(t)}{\partial t}}\right\rangle_{\bar{\alpha}}}^{Ware-Galeev\ pinch} - \frac{1}{\bar{\mathcal{V}}'}\frac{\partial}{\partial\bar{\alpha}}\bar{\mathcal{V}}'\overbrace{\Gamma_1(\bar{\alpha})}^{flux} = \overbrace{U_1(\bar{\alpha})}^{source}$$

Note: *Both* **D** *and* **F** *contribute to flux and source.*

- Flux and source:

$$\begin{bmatrix} \Gamma_1(\bar{\alpha}) \\ U_1(\bar{\alpha}) \end{bmatrix} = \sum_2 \begin{bmatrix} \Gamma_{12}(\bar{\alpha}) \\ U_{12}(\bar{\alpha}) \end{bmatrix} = \sum_2 \sum_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} \sum_{\mathbf{k}} \frac{1}{\mathcal{V}'(\bar{r})} \left(\frac{2\pi}{M_1}\right)^3 \int d\boldsymbol{J}_1 \left(\frac{2\pi}{M_2}\right)^3 \int d\boldsymbol{J}_2$$

$$\overbrace{Q(\boldsymbol{\ell}_1, \boldsymbol{J}_1; \boldsymbol{\ell}_2, \boldsymbol{J}_2)}^{\text{spectrum}} \overbrace{\begin{bmatrix} X^\chi(\boldsymbol{J}_1, t, \boldsymbol{\ell}_1, \bar{\alpha}) \\ Y^\chi(\boldsymbol{J}_1, t, \boldsymbol{\ell}_1, \bar{\alpha}) \end{bmatrix}}^{\text{flux or source}} f_M(\boldsymbol{J}_1) f_M(\boldsymbol{J}_2) \overbrace{\mathcal{A}(\boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2)}^{\text{drives}}$$

where:

$$\begin{bmatrix} X^\chi \\ Y^\chi \end{bmatrix} \equiv \int \frac{d\boldsymbol{\Theta}}{(2\pi)^3} \begin{bmatrix} \chi(\boldsymbol{J}, \boldsymbol{\Theta}, t)\ \boldsymbol{\ell} \cdot \partial_{\mathbf{J}} \alpha(\boldsymbol{J}, \boldsymbol{\Theta}, t) \\ \boldsymbol{\ell} \cdot \partial_{\mathbf{J}} \chi(\boldsymbol{J}, \boldsymbol{\Theta}, t) \end{bmatrix} \delta[\alpha(\boldsymbol{J}, \boldsymbol{\Theta}, t) - \bar{\alpha}]$$

$$Q(\boldsymbol{\ell}_1, \mathbf{J}_1, \boldsymbol{\ell}_2, \mathbf{J}_2) \propto \delta(\boldsymbol{\ell}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{\ell}_2 \cdot \boldsymbol{\Omega}_2)\ \overbrace{C_a(\boldsymbol{\ell}_1, \mathbf{J}_1; \boldsymbol{\ell}_2, \mathbf{J}_2)}^{\text{coupling coeff.}}$$

- The driving term is:

$$
\begin{aligned}
\mathcal{A}(\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{\ell}_1,\boldsymbol{\ell}_2) \equiv\;& \left(1+\frac{V_{\|,1}P_1}{T_1}\right)\left(1+\frac{V_{\|,2}P_2}{T_2}\right)\left(\frac{\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1}{T_1}-\frac{\boldsymbol{\ell}_2\cdot\boldsymbol{\Omega}_2}{T_2}\right)\\
-\;& \left[\left(1+\frac{V_{\|,2}P_2}{T_2}\right)G_1\,\frac{V_{\|,1}}{T_1}-\left(1+\frac{V_{\|,1}P_1}{T_1}\right)G_2\,\frac{V_{\|,2}}{T_2}\right]\\
-\;& \left(1+\frac{V_{\|,1}P_1}{T_1}\right)\left(1+\frac{V_{\|,2}P_2}{T_2}\right)\left(g_1\,\mathcal{A}_{N,1}-g_2\,\mathcal{A}_{N,2}\right)\\
-\;& \left[\frac{V_{\|,1}P_1}{T_1}\left(1+\frac{V_{\|,2}P_2}{T_2}\right)g_1\,\mathcal{A}_{V,1}-\frac{V_{\|,2}P_2}{T_2}\left(1+\frac{V_{\|,1}P_1}{T_1}\right)g_2\,\mathcal{A}_{V,2}\right]\\
-\;& \left(1+\frac{V_{\|,1}P_1}{T_1}\right)\left(1+\frac{V_{\|,2}P_2}{T_2}\right)\left(g_1\,\frac{K_{0,1}}{T_1}\mathcal{A}_{T,1}-g_2\,\frac{K_{0,2}}{T_2}\mathcal{A}_{T,2}\right)\ .
\end{aligned}
$$

$$
g\equiv\boldsymbol{\ell}\cdot\boldsymbol{\nabla}_{\boldsymbol{J}}\alpha(\mathbf{J})\ ,\qquad G\equiv\boldsymbol{\ell}\cdot\boldsymbol{\nabla}_{\boldsymbol{J}}\{Mv_{\|}[\alpha(\mathbf{J})]\}\ ,\qquad K_0\equiv H_0-q\Phi_0=\frac{Mv^2}{2}\ ,
$$

$$
\mathcal{A}_N=N'/N+(q/T)\Phi_0'-(3/2)T'/T\ ,\qquad \mathcal{A}_T=T'/T\ ,\qquad \mathcal{A}_V=V_{\|}'/V_{\|}-T'/T\ .
$$

– Because of $\delta(\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1-\boldsymbol{\ell}_2\cdot\boldsymbol{\Omega}_2)$, $\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1/T_1-\boldsymbol{\ell}_2\cdot\boldsymbol{\Omega}_2/T_2\to 0$ when $T_1=T_2$.

– The drive proportional to G is zero when $V_{\|1},V_{\|2}=0$

SELF-CONSISTENT TRANSPORT IN MAGNETIC TURBULENCE

Evaluation coupling coefficient

- For *passing particles in magnetic turbulence*, the perturbing Hamiltonian reduces to

$$|h_a(\boldsymbol{\ell}, \mathbf{J}; \omega)|^2 \simeq \left|(q/c)v_\parallel A^a_\parallel\right|^2 \delta(\ell_\zeta - n_a) J^2_{\ell_g}(z_g) J^2_{\ell_b - m_a}(z_b)$$

  - $z_{\rm g} = k_\perp \rho_{\rm g}$ and $z_{\rm b} = [(k_{\rm r} r_{\rm d})^2 + (m\theta_{\rm d} + n\zeta_{\rm d})^2]^{1/2}$ [where $(r_{\rm d}, \theta_{\rm d}, \zeta_{\rm d})$ quantify the particle excursion from the field lines in the course of a transit period]
  - $J_\ell(z_g)$, $J_\ell(z_b)$: represent the strength of that portion of mode "a" that is oscillatory at $\exp[i(\ell_g \Theta_g + \ell_b \Theta_b)]$ (i.e., range of ℓ_g and ℓ_b over which a mode contributes)
  - $\delta(\ell_\zeta - n_a)$: gives that portion (namely, all or none) of the mode that is oscillatory at $\exp(i\ell_z \Theta_\zeta)$.

- The spectrum in the collision operator is ***thermal***, i.e., driven solely by *uncorrelated shielded test particles* (as in the standard BL operator)

  To approximate realistic turbulence, *we replace it with a supra-thermal spectrum*, the ***pseudo-thermal ansatz*** of Mynick and Duvall, PoF 1989:

  (i) Assume that the "generalized dielectric function" Δ_a is nonlinearly modified so that ($\Delta k_\perp \sim \rho_{gi}^{-1}$, $\Delta k_\parallel \sim L_s^{-1}$, $\Delta k_\perp \gg \Delta k_\parallel$)

$$\mathbf{A}^a(\mathbf{r}, \omega_a) \to A_\parallel^a \propto \tilde{B}^2(2)\ \exp[-\frac{k_\perp^2}{2(\Delta k_\perp)^2} - \frac{k_\parallel^2}{2(\Delta k_\parallel)^2}]$$

  (ii) Replace the spectrum driven by species 2:

$$\int d\mathbf{J}_2\ f(\mathbf{J}_2) \overbrace{\left|\frac{4\pi\ h_a(\boldsymbol{\ell}_1, \mathbf{J}_1, \omega)\ h_a^*(\boldsymbol{\ell}_2, \mathbf{J}_2, \omega)}{N_a \Delta_a(\omega)}\right|^2}^{C_a} \to \int_{\mathcal{V}_a} d\mathbf{J}_2 f(\mathbf{J}_2) \frac{\left|q_1 A_\parallel^a\right|^2}{\mathcal{V}_a N_2 \langle |u_2/c|^2 \rangle} \left|\frac{u_1 u_2}{c^2}\right|^2$$

Important point: the pseudo-thermal spectrum <u>retains</u> the structure of the original thermal spectrum $\Rightarrow$ it maintains the required properties of the collision operator

Procedure to evaluate fluxes and sources (I)

Procedure consists of three main steps:

$$\begin{bmatrix} \Gamma_1(\bar\alpha) \\ U_1(\bar\alpha) \end{bmatrix} \propto \overbrace{\sum_{\ell_{g1},\ell_{g2}} \sum_{\ell_{\zeta1},\ell_{\zeta2}} \sum_{\ell_{b1},\ell_{b2}}}^{\text{step 1}} \overbrace{\int d\boldsymbol{J}_1 \int d\boldsymbol{J}_2}^{\text{step 2}} \overbrace{\sum_{\mathbf{k}}}^{\text{step 3}}$$

$$\delta(\ell_{\zeta_1} - n_a)\, \delta(\ell_{\zeta_2} - n_a)\ \ \delta(\bar\alpha_1 - \alpha)$$

$$\overbrace{Q(\boldsymbol{\ell}_1, \boldsymbol{J}_1; \boldsymbol{\ell}_2, \boldsymbol{J}_2)}^{\text{spectrum}} \overbrace{\begin{bmatrix} X^\chi(\boldsymbol{J}_1, \boldsymbol{\ell}_1, \bar\alpha) \\ Y^\chi(\boldsymbol{J}_1, \boldsymbol{\ell}_1, \bar\alpha) \end{bmatrix}}^{\text{flux or source}} \overbrace{\mathcal{A}(\boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2)}^{\text{drive}}\ ,$$

where

$$Q \propto \delta(\boldsymbol{\ell}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{\ell}_2 \cdot \boldsymbol{\Omega}_2)\ |A_\parallel|^2\ |v_{\parallel,1}|^2 |v_{\parallel,2}|^2$$

$$\mathcal{A} \propto \boldsymbol{\ell}_j \cdot \boldsymbol{\Omega}_j(\mathbf{J}_j), g_j(\mathbf{J}_j), G_j(\mathbf{J}_j) \qquad j = 1, 2$$

Procedure to evaluate fluxes and sources (II)

Step (1): ℓ sums

The ***sums over $\boldsymbol{\ell}_{\mathrm{g}}$*** are eliminated by setting $\ell_g = 0$ (no gyro-resonances).
The ***sums over $\boldsymbol{\ell}_{\zeta}$*** are performed using $\delta(\ell_\zeta - n_a)$ (axisymmetry).
The ***sums over $\boldsymbol{\ell}_{\mathrm{b}}$*** are converted into integrations (due to resonance broadening effects), and using $\delta(\ell_{b1}\Omega_{b1} + n_a\Omega_{\zeta 1} - \ell_{b2}\Omega_{b2} - n_a\Omega_{\zeta 2})$ in the ℓ_{b1} sum. For example:

$$\int d(\boldsymbol{\ell}_1 \boldsymbol{\cdot} \boldsymbol{\Omega}_1)\ \delta(\boldsymbol{\ell}_1 \boldsymbol{\cdot} \boldsymbol{\Omega}_1 - \boldsymbol{\ell}_2 \boldsymbol{\cdot} \boldsymbol{\Omega}_2) \left(\frac{\boldsymbol{\ell}_1 \boldsymbol{\cdot} \boldsymbol{\Omega}_1}{T_1} - \frac{\boldsymbol{\ell}_2 \boldsymbol{\cdot} \boldsymbol{\Omega}_2}{T_2} \right) \rightarrow (\boldsymbol{\ell}_2 \boldsymbol{\cdot} \boldsymbol{\Omega}_2) \frac{1}{T_2} \left(\frac{T_2}{T_1} - 1 \right)$$

Before proceeding, need expressions of various factors in the far-untrapped limit. For example:

$$\boldsymbol{\ell}_2 \boldsymbol{\cdot} \boldsymbol{\Omega}_2 \xrightarrow{\kappa \to \infty} \omega'_{b2} + k_\parallel v_{\parallel 2} - n_a c \left(\frac{M_2}{q_2} \frac{\partial q_{\mathrm{saf}}}{\partial \alpha_0} v_{\parallel 2}^2 - q_{\mathrm{saf}} \frac{\partial \Phi_0}{\partial \alpha_0} \right)$$

$$X_1^\chi \xrightarrow{\kappa \to \infty} \begin{bmatrix} 1 \\ M_1 v_{\parallel 1} \\ K_2 \end{bmatrix} g_1\ \delta(\alpha - \bar{\alpha})\ , \qquad Y_1^\chi \xrightarrow{\kappa \to \infty} \begin{bmatrix} 0 \\ 1 \\ v_{\parallel 1} \end{bmatrix} G_1\ \delta(\alpha - \bar{\alpha})$$

Step (2): J integrations

For untrapped particles we approximate

$$\int_{\mathcal{V}_a} d\mathbf{J}_j \simeq \frac{M_j^2 R_0}{B_0} \int d\alpha_j \int_{-\infty}^{+\infty} dv_{\|j} \int_{M_j v_{\|j}^2/2}^{+\infty} dK_{0j} \; .$$

The $\mathbf{d\alpha_1}$ *integration* is performed using $\delta(\alpha_1 - \bar{\alpha})$, while the $\mathbf{d\alpha_2}$ *integration* is performed approximating the equilibrium quantities as constants inside $\mathcal{V}_a$. The remaining integrations are trivial.

Step (3): k sums

These sums are converted into integrations,

$$\sum_{\mathbf{k}} = \frac{\mathcal{V}_a}{(2\pi)^2} \int_0^{\infty} dk_\perp \, k_\perp \int_{-\infty}^{\infty} dk_\|$$

and performed in the approximation $k_\| \ll k_\perp$ (magnetic turbulence).

Fluxes

Keeping only terms of $\mathcal{O}(1)$ in $\epsilon \equiv \rho_{\rm e,p}\mathcal{A} \sim 10^{-3}$, where $\mathcal{A}_{\rm N}, \mathcal{A}_{\rm V}, \mathcal{A}_{\rm T}$ or $\mathcal{A}_{\rm saf} \equiv (dq_{\rm saf}/dr)/q_{\rm saf}$, we obtain for the electron particle flux:

$$\Gamma_e^N = 3\mathcal{L}_{ei}\left(\frac{T_i}{T_e} - 1\right)\mathcal{A}_{\rm saf} - \mathcal{L}_{ei}\left(L_{N_e}^{-1} + L_{N_i}^{-1} + L_{T_e}^{-1} + L_{T_i}^{-1}\right)$$

where

$$\mathcal{L}_{12} = \sum_{r_a} p^2 \pi N_1 b_t^2 \hat{D}_{RR}(1,2) \quad \text{with} \quad \hat{D}_{\rm RR}(1,2) = v_{\rm th,1}\frac{v_{\rm th,1}}{\Omega_{\rm b,M}}\frac{v_{\rm th,2}^2}{\langle |v_{\parallel 2}|^2\rangle}\tilde{b}_r^2(2)\frac{J_0^2(z_{\rm g,1})J_0^2(z_{\rm g,2})}{\pi z_{b,M}}$$

- For $(1,2) = (e,i)$, using $v_{\rm th,e}/\Omega_{\rm b,M=e} \simeq q_{\rm saf}R_0$ and $\langle |v_{\parallel i}|^2\rangle = v_{\rm th,i}^2$:

$$\hat{D}_{\rm RR}(e,i) \propto v_{\rm th,e} q_{\rm saf} R_0 \tilde{b}_r^2(i)$$

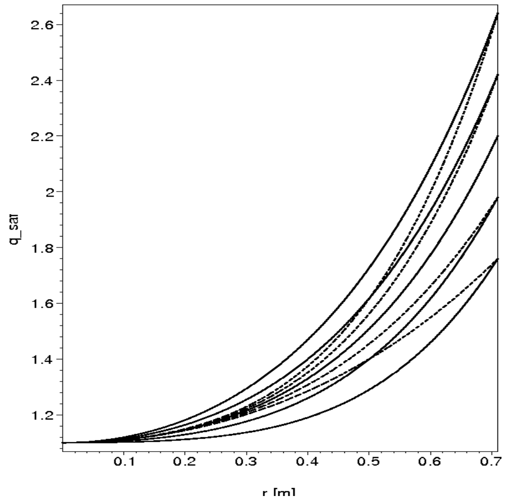
i.e., a *generalized Rechester-Rosenbluth coefficient*

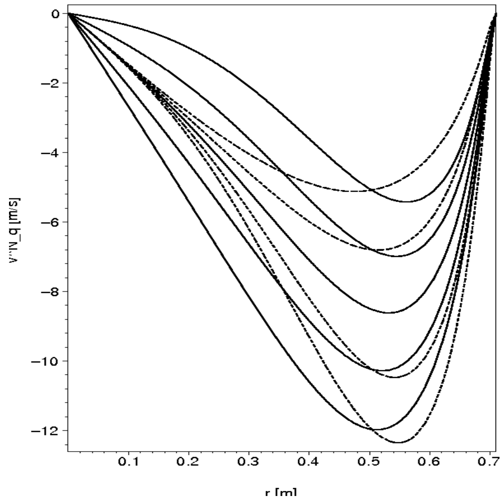
- Electron-electron particle flux is zero, and no dependence on Φ_0
- Compare with conventional expression $\Gamma_e^N = -D_N^N(dN_e/dr) - D_T^N(dT_e/dr) + V_N N_e$:

$$D_N^N = \mathcal{L}_{ei}/N_e \quad \text{diffusion coeff.}\ , D_T^N = \mathcal{L}_{ei}/T_e \quad \text{thermo} - \text{diffusion coeff.}$$

$$V_N = -\frac{\mathcal{L}_{ei}}{N_e}\left[L_{N_i} + L_{T_i} - 3\left(\frac{T_i}{T_e} - 1\right)\mathcal{A}_{q_{\rm saf}}\right] \quad \text{pinch velocity}$$

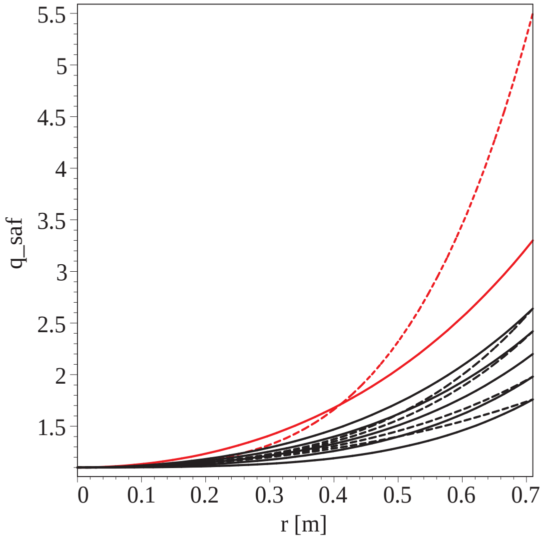
Particle pinch velocity due to $q'_{\rm saf}$

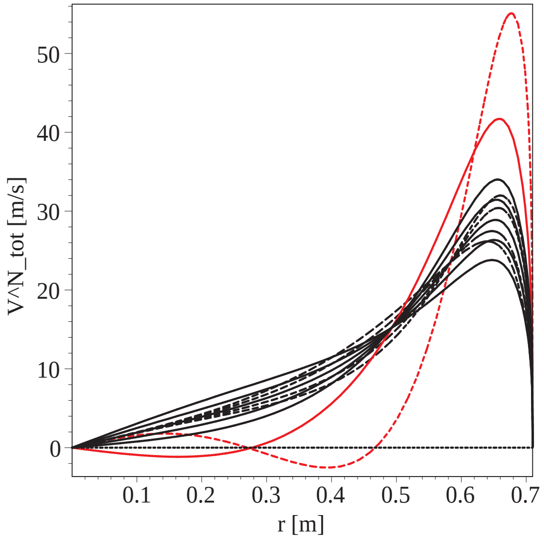




Safety factor profiles (left) and corresponding ***pinch velocity profiles (right)***, for typical tokamak profiles with central values $T_{\rm e,0} = 4.8$ keV and $T_{\rm i,0} = 1.3$ keV, and for $\tilde{b}_r = 1 \times 10^{-4}$.

Total particle pinch velocity





Condition for particle pinch

$$F_{\text{pinch}} \equiv \frac{3|L_{N_i}|\,(1 - T_i/T_e)\,\mathcal{A}_{q_{\text{saf}}}}{1 + \eta_i} > 1 \ ,$$

where $\eta_i \equiv L_{N_i}/L_{T_i}$ and we have assumed conventional discharges with $L_{N_i} = -|L_{N_i}|$, $L_{T_i} = -|L_{T_i}|$, $\eta_i > 0$ and $\mathcal{A}_{q_{\text{saf}}} > 0$

$\Rightarrow$ ***for*** $T_i < T_e$ ***pinch threshold depends on*** $\mathcal{A}_{q_{\text{saf}}}$***,*** T_i/T_e ***and i-profiles***

Keeping only $\mathcal{O}(1)$ terms we obtain for the electron momentum and energy fluxes:

$$\begin{aligned}
\Gamma_e^V &= -3M_eV_{\parallel e}\left\{\mathcal{L}_{ei}\left(L_{N_e}^{-1}+L_{N_i}^{-1}+L_{T_e}^{-1}+L_{T_i}^{-1}\right)+(\mathcal{L}_{ei}+\mathcal{L}_{ee})L_{V_{\parallel e}}^{-1}\right. \\
&\quad \left. -\left[2\mathcal{L}_{ei}\left(\frac{T_i}{T_e}-\frac{3}{2}\right)+\mathcal{L}_{ee}\right]\mathcal{A}_{q_{\rm saf}}\right\} \\
\Gamma_e^T &= -\frac{5}{2}T_e\left\{\mathcal{L}_{ei}\left(L_{N_e}^{-1}+L_{N_i}^{-1}+L_{T_i}^{-1}\right)+(2\mathcal{L}_{ei}+\mathcal{L}_{ee})\,L_{T_e}^{-1}\right. \\
&\quad \left. -3\mathcal{L}_{ei}\left(\frac{T_i}{T_e}-1\right)\mathcal{A}_{q_{\rm saf}}\right\}
\end{aligned}$$

Compare the energy flux with conventional expression $\Gamma_e^T = -\chi_N T_e(dN_e/dr) - \chi_T N_e(dT_e/dr) - \frac{5}{2}\Gamma_e^N T_e + V_T N_e T_e$:

$$\chi_N = 5\frac{\mathcal{L}_{ei}}{N_e}\,, \qquad \chi_T = 5\frac{3\mathcal{L}_{ei}+\mathcal{L}_{ee}}{2N_e} \qquad \text{thermal diffusivities}$$

$$V_T = -5\frac{\mathcal{L}_{ei}}{N_e}\left[L_{N_i}^{-1}+L_{T_i}^{-1}-3\left(\frac{T_i}{T_e}-1\right)\mathcal{A}_{q_{\rm saf}}\right] \qquad \text{pinch velocity}$$

The condition for energy pinch is identical to the condition for particle pinch

Sources

Keeping only $\mathcal{O}(1)$ terms we obtain for the electron momentum and energy sources:

$$(rB_{0t})U_e^V = -3M_eV_{\parallel e}\left[\mathcal{L}_{ei}\frac{T_i}{T_e}\mathcal{A}_{q_{\rm saf}}\left(L_{N_e}^{-1}+L_{N_i}^{-1}+2L_{T_i}^{-1}+5\mathcal{A}_{q_{\rm saf}}\right)\right.$$
$$\left.+\left(\mathcal{L}_{ei}\frac{T_i}{T_e}-\mathcal{L}_{ee}\right)\mathcal{A}_{q_{\rm saf}}L_{V_{\parallel e}}^{-1}+\mathcal{L}_{ee}\left(\frac{\Delta k_\parallel/\Delta k_\perp}{pb_p\rho_{\rm e,p}}\right)^2\right]$$

$$(rB_{0t})U_e^T = -3T_e\mathcal{A}_{q_{\rm saf}}\left[\mathcal{L}_{ei}\frac{T_i}{T_e}\left(L_{N_e}^{-1}+L_{N_i}^{-1}+2L_{T_i}^{-1}\right)+\left(\mathcal{L}_{ei}\frac{T_i}{T_e}+\mathcal{L}_{ee}\right)L_{T_e}^{-1}\right.$$
$$\left.-5\mathcal{L}_{ei}\frac{T_i}{T_e}\mathcal{A}_{q_{\rm saf}}\left(\frac{T_i}{T_e}-1\right)\right]$$

Assuming a true Maxwellian distribution (no N, T gradients):

$$(rB_{0t})U_e^T = 15\mathcal{L}_{ei}T_i\mathcal{A}_{\rm saf}^2\left(T_i/T_e-1\right)$$

$\Rightarrow$ electron-cooling when $T_e > T_i$, as it should be (thanks to friction term!)

Ohm's law

Approximating $V_{\|e} \simeq -j_{\|}/(eN_e)$ in momentum balance, we obtain the ***generalized Ohm's law***

$$-\frac{M_e}{e^2 N_e}\frac{\partial \left\langle j_{\|}\right\rangle_r}{\partial t} + \frac{\left\langle N_e E_t\right\rangle_r}{N_e} = \mathbf{E}_{\|}^{\mathrm{SC}} + E_{\mathrm{BS}} + \eta_{\|}^{\mathrm{neo}} j_{\|}$$

where ***the self-consistent contribution*** is

$$E_{\|}^{\mathrm{SC}} = \eta_{\mathrm{an}} j_{\|} + \eta_{\times}\frac{dj_{\|}}{dr} + \frac{1}{B_0 r}\frac{d}{dr}\left[r\eta_H \frac{d}{dr}\left(\frac{j_{\|}}{B_0}\right)\right]$$

where the order-of-magnitude expressions for the transport coefficients are

$$\eta_H \simeq \frac{4\pi}{\omega_e^2} B_0^2 \frac{\mathcal{L}_{ee}}{N_e}\,, \qquad \eta_\times \simeq \frac{4\pi}{\omega_e^2}\frac{1}{L_{N_e}}\left(1 - \frac{L_{N_e}\hat{\epsilon}}{\rho_{\mathrm{e,p}}}\right)\frac{\mathcal{L}_{ee}}{N_e}\,,$$

$$\eta_{an} \simeq \frac{4\pi}{\omega_e^2}\frac{1}{L_B^2}\left(1 - \frac{L_B^2\hat{\epsilon}}{L_{N_e}\rho_{\mathrm{e,p}}} + \frac{L_B^2\hat{\epsilon}^2}{\rho_{\mathrm{e,p}}^2}\right)\frac{\mathcal{L}_{ee}}{N_e} + \frac{1}{r}\frac{d}{dr}r\left[\frac{4\pi}{\omega_e^2}\frac{1}{L_B}\left(1 - \frac{L_B\hat{\epsilon}}{\rho_{\mathrm{e,p}}}\right)\frac{\mathcal{L}_{ee}}{N_e}\right],$$

where $\omega_e^2 = 4\pi e^2 N_e/M_e$, and $\hat{\epsilon}$ is a nondimensional function of order ϵ.

Transport rates

Compare particle diffusion coefficient D_N^N, (diagonal) thermal diffusivity χ_T, hyper-resistivity η^H and anomalous resistivity η_{an}:

D_N^N	χ_T	$\eta_H/B_0^2 \approx \eta_{\rm an}/(\hat{\epsilon}/\rho_{\rm e,p})^2$
$\mathcal{L}_{ei}/N_e$	$\mathcal{L}_{ee}/N_e$	$(4\pi/\omega_e^2)\mathcal{L}_{ee}/N_e$

- Since $\tilde{b}^2(i)/\tilde{b}^2(e) \propto v_{\rm th,i}^2/v_{\rm th,e}^2 \;\Rightarrow\; \mathcal{L}_{ee} \gg \mathcal{L}_{ei}$, ***particle diffusion is slower than heat diffusion***.
- Because of the small factor $1/\omega_e^2$, where $\omega_e^2 = 4\pi e^2 N_e/M_e$ is the electron plasma frequency, ***current diffusion is slower than heat transport***, $(\eta_H/B_0^2)/\chi_T \sim 4\pi/\omega_e^2 \ll 1$.
- Analogously, $\eta_{\rm an}/\chi_T \sim 4\pi/\omega_e^2 \ll 1$ so that ***anomalous resistivity remains small*** (even though thermal conduction is much enhanced by turbulence):

$$\sigma_{\rm an}/\sigma_{\rm c} \simeq 10^4 \gg 1$$

SUMMARY AND RESULTS

- We have considered the ***action-angle generalization of the Balescu-Lenard collision operator*** to study *electron transport* in *magnetic turbulence*
- The turbulent calculation required to obtain the magnetic spectrum has been avoided by using ***the "pseudo-thermal" spectrum*** of Mynick & Duvall
- We have derived the complete set of ***transport equations*** (particle, momentum, energy and Ohm's law) for passing electrons
- We have found:

  (i) ***Drives proportional to*** q_s'/q_s are present in all transport fluxes and sources

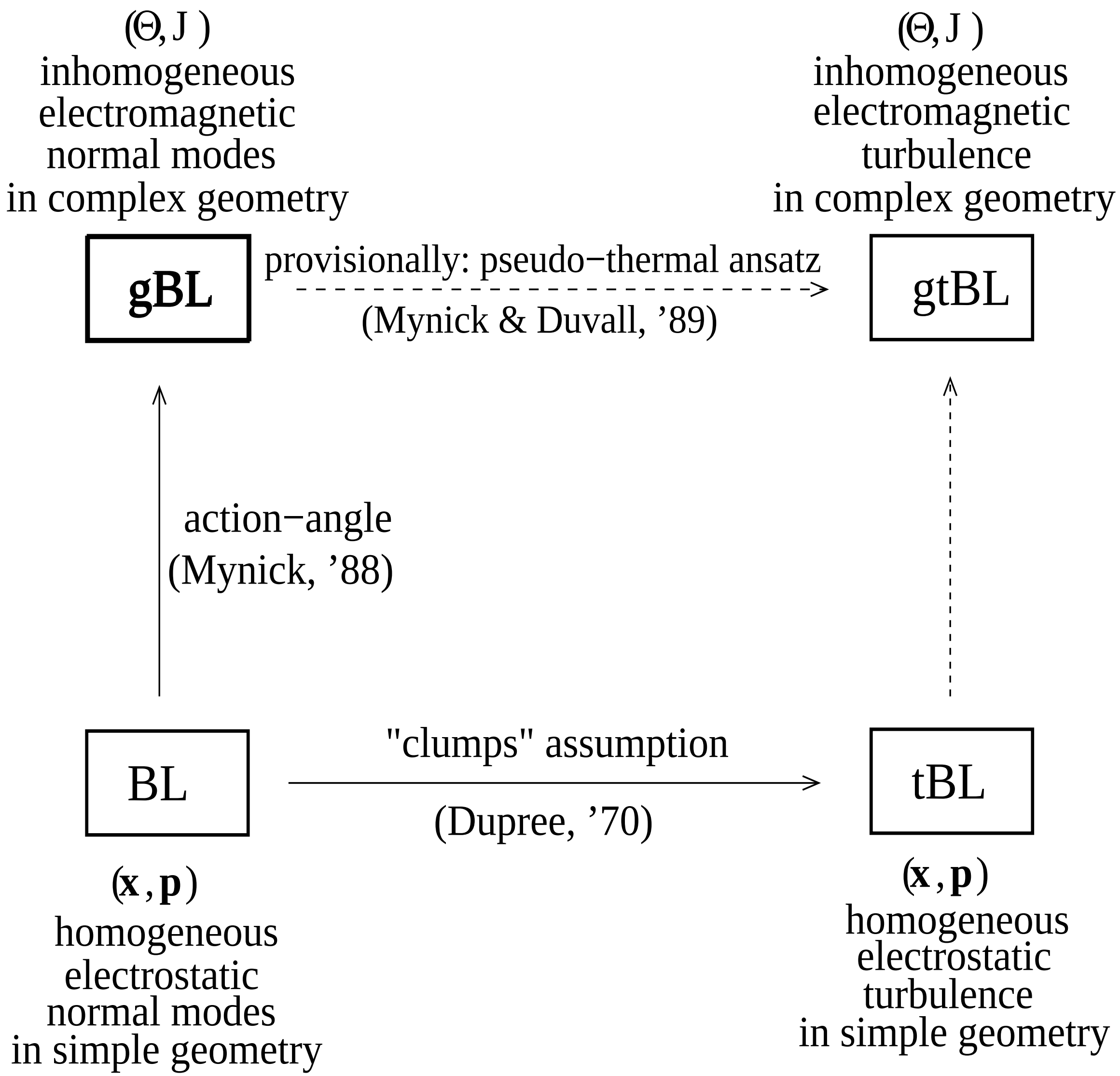
  (ii) Particle and energy pinches "driven" by q_s'/q_s ***depend crucially on*** T_i/T_e, and on $\eta_i \equiv L_{N_i}/L_{T_i}$

  (iii) (hyper-resistive) *current diffusion* slower than *particle diffusion* slower than *heat diffusion*

  (iv) ***Anomalous resistivity very small***

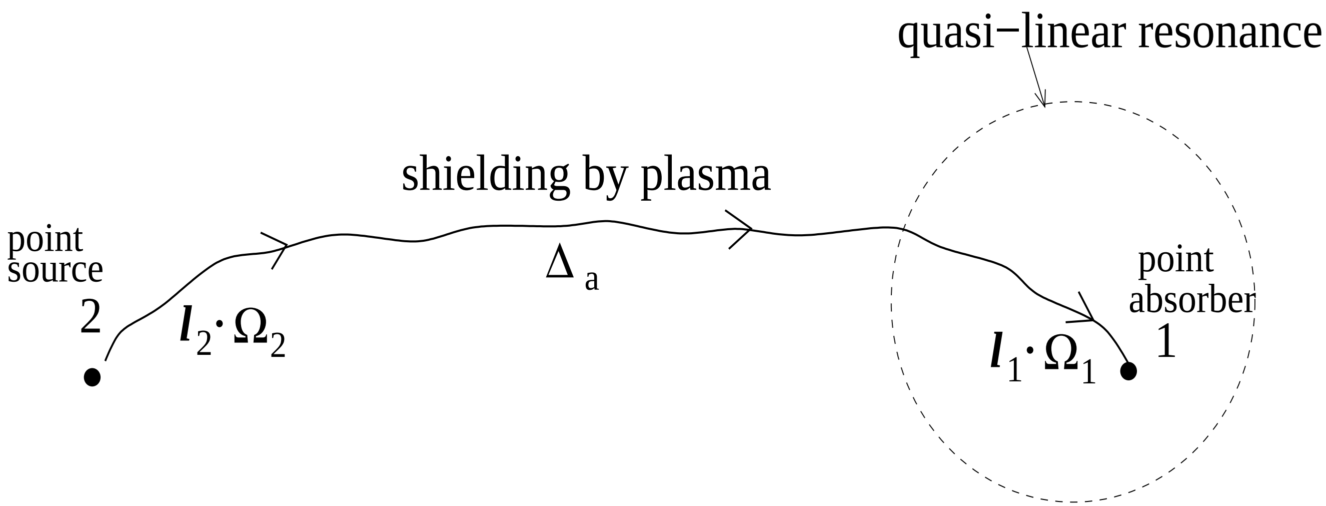
FUTURE PLANS

- Carry out similar calculation for trapped particles in *electrostatic turbulence (TEM)*
- Improve on the "pseudo-thermal" ansatz. This requires the derivation of a *turbulent version of the generalized BL operator*. A possible model is Dupree turbulent version of the BL operator
  - The BL includes the *collective nature* of the plasma dynamics and describes long-range interactions by the electrostatic field
  - In some cases the *initial conditions (the ballistic term)* in the propagator of the correlation can affect the relaxation, especially if the initial conditions deviate greatly from the equilibrium distribution
  - The ballistic effect is usually analyzed in terms of *clumps*, i.e., clusters of particles moving together and thus forming a macroparticle with a finite life time (Dupree; Terry, Diamond and Hahm)



(Ref: Mynick, APS 1989)

Physical interpretation of self-consistent spectrum



$$\delta(\Theta-\Theta_2) = \sum_{\boldsymbol{l}_2} |\boldsymbol{l}_2\rangle \longrightarrow \sum_{a,\boldsymbol{l}_2} \frac{|a\rangle \langle a|\boldsymbol{l}_2\rangle}{N_a \, \Delta_a} \longrightarrow \sum_{a,\boldsymbol{l}_1,\boldsymbol{l}_2} \frac{\langle \boldsymbol{l}_1|a\rangle \langle a|\boldsymbol{l}_2\rangle}{N_a \, \Delta_a}$$

$$\boxed{\frac{h(\boldsymbol{l}_1, a)\, h^*(\boldsymbol{l}_2, a)}{N_a \, \Delta_a}}$$

coupling coefficient

(Ref: Mynick, APS 1989)

Parallel with standard Balescu-Lenard operator

$$\frac{\partial f(\mathbf{v}_1;t)}{\partial t} = \frac{\partial}{\partial \mathbf{v}_1} \cdot \left[\mathbf{D}^{BL}(\mathbf{v}_1) \cdot \frac{\partial f(\mathbf{v}_1;t)}{\partial \mathbf{v}_1} - \mathbf{F}^{BL}(\mathbf{v}_1) f(\mathbf{v}_1;t) \right]$$

$$\frac{\partial f(\mathbf{J}_1;t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[\mathbf{D}^{gBL}(\mathbf{J}_1) \cdot \frac{\partial f(\mathbf{J}_1;t)}{\partial \mathbf{J}_1} - \mathbf{F}^{gBL}(\mathbf{J}_1) f(\mathbf{J}_1;t) \right]$$

$$\begin{bmatrix} \mathbf{D}^{BL}(\mathbf{v}_1) \\ \mathbf{F}^{BL}(\mathbf{v}_1) \end{bmatrix} = \int d\mathbf{k}\, \mathbf{k} \left(\frac{2\pi}{M_2} \right)^3 n_0 \int d\mathbf{v}_2\, Q^{BL}(\mathbf{k}, \mathbf{v}_1; \mathbf{k}, \mathbf{v}_2) \begin{bmatrix} \mathbf{k} f(\mathbf{v}_2;t) \\ \mathbf{k} \cdot \partial_{\mathbf{v}_2} f(\mathbf{v}_2;t) \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{D}^{gBL}(\mathbf{J}_1) \\ \mathbf{F}^{gBL}(\mathbf{J}_1) \end{bmatrix} = \sum_a \sum_{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2} \boldsymbol{\ell}_1 \left(\frac{2\pi}{M_2} \right)^3 \int d\mathbf{J}_2\, Q^{gBL}(\boldsymbol{\ell}_1, \mathbf{J}_1; \boldsymbol{\ell}_2, \mathbf{J}_2) \begin{bmatrix} \boldsymbol{\ell}_1\, f(\mathbf{J}_2;t) \\ \boldsymbol{\ell}_2 \cdot \partial_{\mathbf{J}_2} f(\mathbf{J}_2;t) \end{bmatrix}$$

$$Q^{BL}(\mathbf{v}_1, \mathbf{v}_2) = 2\pi\, \delta(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2) \left| C^{BL}(\mathbf{k}, \mathbf{v}_1) \right|^2$$

$$Q^{gBL}(\boldsymbol{\ell}_1, \mathbf{J}_1, \boldsymbol{\ell}_2, \mathbf{J}_2) = 2\pi\, \delta(\boldsymbol{\ell}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{\ell}_2 \cdot \boldsymbol{\Omega}_2) \left| C^{gBL}(\boldsymbol{\ell}_1, \mathbf{J}_1, \boldsymbol{\ell}_2, \mathbf{J}_2, \omega_a) \right|^2$$

$$C^{BL}(\mathbf{k}, \mathbf{v}_1) = \frac{\phi(\mathbf{k})}{\varepsilon(\mathbf{k} \cdot \mathbf{v}_1)}$$

$$C^{gBL}(\boldsymbol{\ell}_1, \mathbf{J}_1, \boldsymbol{\ell}_2, \mathbf{J}_2, \omega_a) = \frac{4\pi\, h_a(\boldsymbol{\ell}_1, \mathbf{J}_1, \omega)\, h_a^*(\boldsymbol{\ell}_2, \mathbf{J}_2, \omega)}{N_a \Delta_a(\omega)}$$