Nonlinear Refractive Suppression of Turbulence and Transport by Strong Magnetic Shear

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Outline

Transport barriers:
   Fusion plasmas
   Turbulence intermittency
Intermittency in interstellar turbulence
Shear Alfvén and kinetic Alfvén wave cascades
Refractive suppression of turbulence mixing in KAW turbulence
   Mixing of current
   Mixing of density
Modeling probability distribution functions
Simulation results
Discussion
Transport barriers are essential for improved confinement in fusion

H mode

Edge barrier tied to a region of very strong shear flow
Suppression of turbulence by shear flow widely studied

Nonlinear: turbulent eddies
driven by $\nabla T$, $\nabla n$ sheared apart

\[ \omega_{\text{shear}} > (\tau_{\text{correlation}})^{-1} \]

Linear: shear stabilizes $\nabla T$, $\nabla n$
driven instabilities

\[ \omega_{\text{shear}} > \gamma_{\text{linear}} \]

Internal transport barriers
Suppression by combination of flow shear (as above) and reversed magnetic shear

Magnetic shear understood as linear stabilization mechanism
Magnetic shear also suppresses via nonlinear mechanism

Linear mechanism:
  Impose sheared magnetic equilibrium - turn crank - get growth rate
  Growth rate sensitive to profiles ⇒ repeat for every new profile

Nonlinear mechanism:
  Not sensitive to profile ⇒ more universal
  Mathematically equivalent to shear flow mechanism:
    \[ \omega_{shear} > (\tau_{correlation})^{-1} \]
  Physically very different
    No flow
    No shearing a la cartoon

Explore mechanism in context of turbulence intermittency:
  Transport suppression allows special fluctuations to be coherent
Two types of intermittency explained by two turbulence suppression mechanisms (flow shear, magnetic shear)

Flow shear ↔ Coherent vortices in decaying 2D Navier-Stokes turbulence

Simulations: long lived vortices emerge from Gaussian initial pdf (McWillaims, JFM 84)

High vorticity structures live forever; average vorticity structures die

Theory: coherent vortices have edge shear flow ⇒ barrier to mixing (Terry, 89)

Nonlinear suppression mechanism $\omega_{\text{shear}} > (\tau_{\text{correlation}})^{-1}$ maps onto structure profiles in Gaussian curvature (GC)

- $\text{GC} \ll 1$  strong vortex, no shear, no turbulence
- $\text{GC} \gg 1$  strong vortex, strong shear, no mixing
- $\text{GC} \approx 1$  vorticity, shear balanced ⇒ turbulent region with strong mixing
Magnetic shear ↔ Coherent current filaments in decaying kinetic Alfvén wave turbulence

Simulation: long lived current filament emerge from Gaussian initial pdf (Craddock 91)

High current structures live forever; average current structures die

Filaments strikingly similar to vortices of 2D Navier Stokes

What quantity has strong edge shear?

Answer: magnetic field

Does magnetic shear do to KAW turbulence what flow shear does to NS turbulence?
Magnetic shear suppresses mixing by refraction of KAW turbulence \(\Rightarrow\) coherent density filaments form in interstellar turbulence

Intermittency in interstellar turbulence:
Pulsar scintillation scaling \(\Rightarrow\) intermittent electron density in ISM (Boldyrev 05)

Pulsar signal dispersed by electron density fluctuations in ISM
Pulse width scales as \(R^4\) (\(R\): distance from source)
If pdf of electron density fluctuations Gaussian, signal width \(\sim R^2\)
If pdf is Levy distributed (long tail), can recover \(R^4\) scaling

Key questions:
- Does current intermittency of KAW turbulence support coherent density structures?
- How does it all work?
- How can structures form without flow?
  (all prior intermittency mechanisms involve flow)
- What differentiates structures from surrounding flow?
**Shear Alfvén and kinetic Alfvén cascades**

How does electron density evolve in magnetic turbulence?

**Basic model: RMHD + compressible electron continuity**

\[
\frac{\partial \hat{\psi}}{\partial t} + \nabla \cdot \hat{\phi} = \eta \hat{J} + \nabla \cdot \nabla \hat{n} + \frac{C_s}{V_A} n_0^{-1} \nabla \cdot (\psi \times z \cdot \nabla n_0), \tag{1}
\]

\[
\frac{\partial}{\partial t} \nabla^2 \phi - \nabla \times (\nabla \nabla^2 \phi) = -\nabla \cdot \hat{J}, \tag{2}
\]

where

\[
\nabla_{\parallel} = \frac{\partial}{\partial z} + \nabla \psi \times z \cdot \nabla,
\]

\[
\hat{J} = \nabla_{\perp}^2 \hat{\psi},
\]

\[
\frac{\partial \hat{\phi}}{\partial t} - \nabla \phi \times z \cdot \nabla \hat{n} + \nabla_{\parallel} \hat{J} - \frac{C_s}{V_A} n_0^{-1} \nabla \phi \times z \cdot \nabla n_0 = 0, \tag{3}
\]

<table>
<thead>
<tr>
<th>Field</th>
<th>Term (large scale) Turbulent Alfvén wave</th>
<th>Term (small scale) Kinetic Alfvén wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron density</td>
<td>Elec. advection (⊥ flux): v·∇n</td>
<td>Il compr. (Il flux):B ·∇J</td>
</tr>
<tr>
<td>Flow</td>
<td>Lorentz force: B ·∇J</td>
<td>Ion advection: v·∇v</td>
</tr>
<tr>
<td>Magnetic field</td>
<td>Parallel electric field: B ·∇φ</td>
<td>Elec. pressure: B ·∇ n</td>
</tr>
</tbody>
</table>
Shear Alfvén and kinetic Alfvén cascades

Fluctuations change character across the gyroradius scale (Fernandez et al.)

Large scales \((k\rho_i << 1)\): Alfvén waves
- Coupling: B and \(v\)
- Density: advected passively (no reaction back on B or \(v\))
- Intermittency: Primary structure is current filament
  - Ancillary structure in vorticity
  - Density tracks flow; flow is integral of vorticity
    \(\Rightarrow\) density not strongly intermittent

Small scales \((k\rho_i > 0.1 \text{ in ISM})\): Kinetic Alfvén waves
- Coupling: B and \(n\)
- Flow: Dominated by self advection, decouples from B and \(n\)
- Intermittency: Under study
Shear Alfvén and kinetic Alfvén cascades

Electron density fluctuations increase in kinetic Alfvén regime as $n$ equipartitions with $B$

Spectral energy transfer:

$k \rho_i \ll 1$: transfer dominated by $v \leftrightarrow B$

$k \rho_i \geq 1$: transfer dominated by $n \leftrightarrow B$

Low $k$:

$v$ and $B$ equipartitioned

Density at level dictated by $\nabla \cdot (v \nabla n_0)$

High $k$:

$n$ and $B$ equipartitioned, even if no linear or external drive of density

$v$ and $B$ decouple

Low $k$ - high $k$ crossover at $k \rho_i < 1$
Refractive suppression of mixing in KAW turbulence

Kinetic Alfvén wave turbulence modeled by reduced fluid system for $B$ and $n$

$$\frac{\partial \psi}{\partial t} - \mathbf{B} \cdot \nabla n = \eta \nabla^2 \psi$$ \hspace{1cm} \text{Ohms law for flux, with electron pressure force}

$$\frac{\partial n}{\partial t} + \mathbf{B} \cdot \nabla \nabla^2 \psi = \nu \nabla^2 n$$ \hspace{1cm} \text{Density continuity with parallel compression}

where

$$\mathbf{B} = B_0 \hat{z} + \nabla \psi \times \hat{z}$$ \hspace{1cm} \text{Field,} \hspace{1cm} \mathbf{J} = \nabla^2 \psi \hat{z} \hspace{1cm} \text{Current}

Fluctuations in plane perpendicular to guide field (like reduced MHD)
No ion flow; ions are fixed, neutralizing background
Regimes analogous to low ($\nu < \eta$), high ($\nu > \eta$) magnetic Prandtl number
govern temporal decay
Refractive suppression of mixing in KAW turbulence

Kinetic Alfvén waves propagate along guide, filament fields

Along guide field

\[ \omega = k_z k \quad \leftarrow \text{wavenumber perpendicular to guide field} \]

\[ \uparrow \text{wavenumber along guide field} \]

Fluctuating field, density equipartition:

\[ -k \psi_k = i b_k = n_k \]

Along field of a current filament (isolate by setting \( k_z = 0 \))

\[ \omega = \frac{B_\theta(k_f)}{B_0} k_\theta k = \frac{B_\theta(k_f)}{B_0} k_\theta (k_\theta^2 + k_r^2) \]

Dispersion depends only on wavenumbers \( \perp \) to guide field

To isolate refraction of KAW by filament field, set \( k_z = 0 \)
Filament, turbulence evolve on two time scales

\[
\psi = \psi_0(r, \tau) + \tilde{\psi}(r, \theta, t) \quad \leftarrow \text{KAW turbulence: evolves on rapid time } t, \text{ has variation in } \theta
\]

\[
n = n_0(r, \tau) + \tilde{n}(r, \theta, t)
\]

↑Coherent structure: evolves on slow time \( \tau \), uniform in \( \theta \)

Assume there is density component - see if it survives mixing

Filament localized

- Structure is nonuniform, stationary refracting medium for KAW turbulence
- Turbulence mixes structure by turbulent stresses (averaged over \( t \))
- Seek conditions under which mixing of structure becomes very slow
- Expect slow mixing when refraction of KAW turbulence by structure becomes very strong \( \Rightarrow \) shear in \( B_\theta \) very strong
- Localization of \( J_0 \) \( \Rightarrow \) \( B_\theta \) shear zero in filament core, strong in filament edge
Refraction suppression of mixing in KAW turbulence

Mixing suppression in KAW turbulence similar to 2D NS (but different fields are involved and meaning changes)

No flow, but localized $J$ of coherent structure creates inhomogeneous $B$ ($V_A$) that refracts turbulence away from structure

<table>
<thead>
<tr>
<th>KAW</th>
<th>2D Navier Stokes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Localized coherent structure (origin at center of structure)</td>
<td>$J_z(r) = \mu_0(\nabla \times B)_z$ (current)</td>
</tr>
<tr>
<td>Inhomogeneous azimuthal “flow” of structure</td>
<td>$B_\theta(r) \rightarrow V_A(r)$</td>
</tr>
<tr>
<td>Turbulence</td>
<td>Kinetic Alfven waves $\tilde{b}<em>r, \tilde{b}</em>\theta, \tilde{n}$</td>
</tr>
<tr>
<td>Turbulence source</td>
<td>$n_0(r), J_z(r)$ (structure)</td>
</tr>
<tr>
<td>Agent that prevents mixing</td>
<td>Inhomogeneity of $B_\theta$ refracts KAW turbulence</td>
</tr>
</tbody>
</table>
Refractive suppression of mixing in KAW turbulence

Slow time evolution: stresses of rapidly evolving KAW turbulence mix structure

\[
\frac{\partial n_0}{\partial \tau} + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d \gamma'}{d \gamma} \sum_{m'} \left( \left[ \bar{b}_r(-m',-\gamma') \frac{\partial}{\partial r} + \bar{b}_\theta(-m',-\gamma') \frac{im'}{r} \right] \nabla^2 \tilde{\psi}(m',\gamma',r) \right) = \nu \nabla^2 n_0
\]

\[
\frac{\partial \psi_0}{\partial \tau} - \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d \gamma'}{d \gamma} \sum_{m'} \left( \left[ \bar{b}_r(-m',-\gamma') \frac{\partial}{\partial r} + \bar{b}_r(-m',-\gamma') \frac{im'}{r} \right] \tilde{n}(m',\gamma') \right) = \eta \nabla^2 \psi_0
\]

↑Turbulent stresses

Turbulence: refracted by magnetic shear of filament

\[
\gamma \tilde{\psi}_{\gamma,m} + \mathbf{B}(r) \cdot \nabla \tilde{n}_{\gamma,m} + \tilde{\mathbf{b}} \cdot \nabla \tilde{n} + \tilde{\mathbf{b}} \cdot \nabla n_0(r) = 0
\]

\[
\gamma \tilde{n}_{\gamma,m} + \mathbf{B}(r) \cdot \nabla \tilde{\mathbf{j}}_{\gamma,m} + \tilde{\mathbf{b}} \cdot \nabla \tilde{\mathbf{j}} + \tilde{\mathbf{b}} \cdot \nabla j_0(r) = 0
\]

↑Refraction: KAW propagation in sheared structure field
Refractive suppression of mixing in KAW turbulence

Large magnetic shear $\Rightarrow$ strong refraction of turbulence away from filament core

Refraction in cylindrical field governed by shear:

$$j' = \frac{d}{dr}\left[\frac{B_\theta(r)}{r}\right]$$

$j'$ measures distortion of KAW phase fronts from linear rays

For $B_\theta \sim r$ rays are linear

Expand $B_\theta/r$ in Taylor series about edge location $r_0$

When $j'$ large, system remains turbulent only if nonlinearity balances shearing $\Rightarrow$ classic asymptotic boundary layer of small width $\Delta r$ when $j'$ large:

$$m\Delta r \frac{\partial}{\partial r}\left(\frac{B_\theta}{r}\right)_{r_0} \approx \tilde{b}_r \frac{1}{\Delta r} \tilde{n} \quad \Rightarrow \quad \Delta r^2 = \frac{\tilde{b}_r}{mj'}$$

$\Delta r \ll a$ when $j'$ large

$\Rightarrow B_\theta \gg \tilde{b}_r$
Refractive suppression of mixing in KAW turbulence

Asymptotic boundary layer analysis yields spatial, temporal scales of turbulence in boundary layer

Boundary layer must properly account for KAW dynamics
Closure theory keeps track of KAW dynamics
Closure theory is Gaussian - apply because Gaussian-breaking shearing field is stationary on fast scale - makes turbulence inhomogeneous

Non Gaussian features develop on slow scale
Closure:

• Complicated: 6 separate diffusivities of differing orders
• Enter in same order when paired with corresponding derivatives
• System is 8th order PDE in $r$, not amenable to WKB
• Treat with dimensional analysis $\Rightarrow$ find spatial, temporal scales in terms of parameters; functional forms of variation not found

Dimensional analysis $\Rightarrow \frac{\partial}{\partial r} \rightarrow 1/\Delta r$, PDE $\rightarrow$ algebraic equation
Refraactive suppression of mixing in KAW turbulence

Dimensional solutions of closure equations: refraction limits turbulence to filament edge

Apply boundary layer asymptotic analysis:

- \( j \spcn \) formally ordered large
- \( \Delta r \) formally order small

Unique functional relationship yields consistent balance

\[
\frac{1}{(\Delta r)^2} = \frac{(imj'\hat{d}_2 - \gamma \hat{d}_3)}{2\hat{d}_1^2} + \frac{1}{2\hat{d}_1^2} \left[ (imj'\hat{d}_2 - \gamma \hat{d}_3)^2 - 4\hat{d}_1^2(m^2 j'' + \gamma^2) \right]^{1/2}
\]

\( \gamma \): turbulence decay rate in boundary layer - order \( j \spcn \)

\( \hat{d}_n \): Closure diffusivities - all of same order with \( \hat{d}_n \sim \frac{\bar{\psi}}{a} \)

Layer width narrow when shear large:

\[
\Delta r \sim \hat{d}_n \sqrt{\frac{m j''}{r}}
\]

Layer at edge in strong shear region
Refractive suppression of mixing in KAW turbulence

Dimensional solutions of closure equations: refraction is strong when filament field is large compared with surrounding values

2. Condition for strong refraction

\[
\frac{\Delta r}{a} \sim \sqrt{\frac{d|_{r > a}}{a^2 m j'}} \sim \sqrt{\frac{\tilde{\psi}|_{r > a}}{a^3 m j'}} \ll 1
\]

Magnetic analog of BDT flow shear suppression criterion

since

\[
j' = \frac{d}{dr} \left( \frac{B_\theta}{r} \right) \approx \frac{B_\theta}{a^2}
\]

and

\[
\frac{\tilde{\psi}|_{r > a}}{a} \approx \tilde{b}_\theta |_{rms}
\]

\[
\frac{\Delta r}{a} \sim \sqrt{\frac{\tilde{b}_\theta |_{rms}}{B_\theta}}
\]

⇒ Refraction strong, refractive boundary layer small when filament field is significantly larger than rms turbulent field
Refractive suppression of mixing in KAW turbulence

Dimensional solutions of closure equations: refraction enhances turbulent decay in filament

3. Decay rate of turbulence in filament

Balance of refraction rate, turbulent diffusion rate ⇒ fast decay

\[ \gamma \sim imj' \]

Compared to \( \tau_{turb} \sim a^2/|\tilde{b}_\theta|_{rms} \) (turbulent decay time outside filament):

\[ \gamma\tau_{turb} \sim \frac{imj'}{|\tilde{b}_\theta|_{rms}/a^2} \sim \frac{B_\theta}{|\tilde{b}_\theta|_{rms}} \gg 1 \]

⇒ Turbulence in filament decays rapidly relative to turbulence outside
Refractive suppression of mixing in KAW turbulence

Filament decay time from Reynolds stress-like turbulent correlations is very small

Evaluate stresses like asymptotic dimensional analysis:

$$\int_{-i\infty}^{i\infty} \gamma' \sum_{m'} \left[ \tilde{b}_r(-m',-\gamma') \frac{\partial}{\partial r} + \tilde{b}_\theta(-m',-\gamma') \frac{im'}{r} \right] \nabla^2 \tilde{\psi}(m',\gamma',r)$$

$$\Rightarrow \quad \tau_n' = \tau_{\nabla^2 \tilde{\psi}}' = \left( \frac{a}{\Delta r} \right)^3 \sim \left( \frac{B_\theta}{\tilde{b}_\theta |_{\text{rms}}} \right)^{3/2}$$

Boundary layer mixing time slow compared to turbulent decay time

$$\tau_n / \tau_{\text{turb}} = \tau_{\nabla^2 \tilde{\psi}} / \tau_{\text{turb}} = \frac{a}{\Delta r} \sim \left( \frac{B_\theta}{\tilde{b}_\theta |_{\text{rms}}} \right)^{1/2}$$

Mixing time slow compared to external turbulence times

By time mixes across a few layer widths, turbulence has decayed away

$$\Rightarrow \text{Filament lifetimes arbitrarily large}$$
Refractive suppression implies characteristic profile for Gaussian curvature (for comparison with simulations)

Gaussian curvature is a topological construct

For total magnetic field (filament + turbulence) in KAW:

\[ C_T = \left[ \frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} \right]^2 + \left[ \frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} \right]^2 - \left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]^2 \]

Refractive suppression

Filament core: \( J_0 \) is largest field; \( d/dr(B_0) \to 0 \) \( \Rightarrow C_T \) large and negative

Filament edge: \( d/dr(B_0) \) is largest field; \( J_0 \to 0 \) \( \Rightarrow C_T \) large and positive

Outside filament: \( J_0 \) and \( B_\theta \) small; stresses, current balance

\( \Rightarrow C_T \) near zero everywhere

This profile is observed in simulations
Simulation

Gaussian curvature is zero except at location of filaments

Gaussian curvature is negative in filament core, positive in edge
True for filaments of either sign
Strongly confirms that intermittency due to refraction of KAW turbulence in large shear of filament
Probability distribution functions become non Gaussian as turbulence decays

Current fluctuations above a critical value of current do not decay, everything else does

\( J < J_C \) : PDF collapses onto \( \delta(J) \)

\( J > J_C \) : No evolution

\( \Rightarrow \) greatly enhanced tail

Kurtosis (related to 2nd order moment)

Gaussian: \( \kappa = 3 \)

Current PDF: \[ \kappa = \frac{3}{2} \left( \frac{l}{a} \right)^2 \left[ 1 + O \left( \frac{j_{\text{rms}}(t = 0)}{J_C} \right) \right] \gg 3 \]

\( l \) is mean filament separation
Density is also non Gaussian, but kurtosis need not be large

Current is localized
Magnetic field extends beyond filament with $1/r$ falloff
In KAW, density and magnetic field in equipartition
$\Rightarrow$ density extends beyond filament with $1/r$ falloff

With $n \propto 1/r$ and $P(n)dn = 2\pi rdr$

the PDF, after several decay times, is $P(n) \equiv C_n \frac{1}{n^3}$

Pulsar rf waves are scattered by density gradient with $\nabla n \propto 1/n^2$

$\Rightarrow P(\nabla n) = C_{\nabla n} \frac{1}{n^2}$

Levy distribution

Consistent with inferences from pulsar width scaling
Simulation

With $\eta = 0$ current evolves from initial Gaussian state to highly intermittent state

Filaments are circular
Kurtosis reaches $\sim 10^2$
Density kurtosis remains close to 3
Density filaments develop in region of current filaments when $\nu = 0$

Simulation

1 to 1 mapping from current to density filaments
Density structures are less pronounced - consistent with non localization
Conclusions

Shear-flow induced transport barriers (H mode) have magnetic analog

Strong magnetic field inhomogeneity refracts turbulence if it is amenable to refraction. i.e., if turbulence is wavelike and sensitive to B field

Kinetic Alfvén wave turbulence is example
  Refraction explains observation of coherent current filaments in simulations of KAW turbulence

Theoretical calculation shows refractive effect
  Predicts observed spatial behavior of Gaussian curvature
  Predicts Levy distribution for density gradient
  Consistent with observations

Other kinds of strong inhomogeneities that don’t drive instability may be capable of forming transport barriers