Nonlinear excitation and damping of Zonal Flows using a renormalized polarization response

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Motivation

- Zonal Flows suppress ITG turbulence
- They are not damped by collisionless linear mechanisms (Landau damping)
- The mechanism for the nonlinear damping of Zonal Flows is therefore an important subject for research
- Other suggested mechanisms, e.g. the tertiary instability, do not seem to work
- We suggest another approach to understanding the damping mechanism: the renormalized polarization response
Outline

• Review of the Zonal Flow polarization calculation
• Consider a simpler example: dressed test-particles in gyrokinetic plasmas
• Renormalized dielectric response from coherent mode-coupling
• Potential fluctuation spectrum from renormalized dielectric response to particle discreteness (noise)
• Conclusions
Zonal Flows

Drift-kinetic equation for the nonadiabatic part of the ion distribution function:

\[
\frac{\partial g_{\tilde{q}}}{\partial t} + v_{\parallel} \hat{b} \cdot \nabla g_{\tilde{q}} + i\tilde{q} \cdot \vec{v}_d g_{\tilde{q}} = \frac{e}{T_i} F_i \frac{\partial \phi_{\tilde{q}}}{\partial t} - \frac{1}{B} [\phi, g]_{\tilde{q}}
\]  

(1)

where \( \tilde{q} = (0, 0, q) \), and where the \( \vec{E} \times \vec{B} \) nonlinearity has been written as

\[
[\phi, g] \equiv \hat{b} \times \nabla \phi \cdot \nabla g
\]  

(2)

Assume ion bounce time is shortest time scale, so

\[
g_{\tilde{q}} = \exp(-iQ)h_{\tilde{q}}
\]  

(3)

where \( Q = qv_{\parallel}/\Omega_p \) and \( \hat{b} \cdot \nabla h_{\tilde{q}} = 0 \). After bounce averaging,

\[
\frac{\partial h_{\tilde{q}}}{\partial t} = \frac{e}{T_i} F_i(e^{iQ}) \frac{\partial \phi_{\tilde{q}}}{\partial t} - \left( e^{iQ} \frac{1}{B} [\phi, g]_{\tilde{q}} \right)
\]  

(4)
A simpler problem: dressed test-particles in gyrokinetic plasmas

Ion drift-kinetic equation, unsheared slab model:

\[
\left( \frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial z} \right) f - \frac{e}{m_i} \left( \frac{\partial \phi}{\partial z} \frac{\partial F}{\partial v_\parallel} + \frac{1}{\Omega_i} \frac{\partial \phi}{\partial y} \frac{\partial F}{\partial x} \right) = -\frac{1}{B} \mathbf{\hat{z}} \times \nabla \phi \cdot \nabla f
\]

(5)

Initial conditions:

\[
f(\mathbf{x}, v_\parallel, 0) = \sum_i w_{i0} \delta(\mathbf{x} - \mathbf{x}_{i0}) \delta(v_\parallel - v_\parallel i0)
\]

(6)

Quasineutrality with Boltzmann electrons:

\[
\frac{n_oe^2}{T_e} \phi = \frac{n_oe^2}{T_i} \rho_i^2 \nabla^2 \phi + e \int dv_\parallel f
\]

(7)
Fourier-Laplace transform:

\[
(-i\omega + ik_z v_\parallel) f_{\vec{k},\omega} - \mathcal{L}_{\vec{k}} \phi_{\vec{k},\omega} = f_{\vec{k}}(v_\parallel, 0) + \mathcal{N}_{\vec{k},\omega}[\phi, f]
\]  

(8)

where \( p = -i\omega \) is the Laplace transform parameter, the linear operator is

\[
\mathcal{L}_{\vec{k}} = \frac{ie}{m} \left( k_z \frac{\partial F}{\partial v_\parallel} + \frac{k_y}{\Omega} \frac{\partial F}{\partial x} \right)
\]  

(9)

and the ExB nonlinearity is

\[
\mathcal{N}_{\vec{k},\omega}[\phi, f] = \frac{i}{B} \sum_{k',\omega'} (\vec{k} \cdot \vec{k}' \times \hat{z}) \phi_{\vec{k}',\omega'} f_{\vec{k}-\vec{k}',\omega-\omega'}
\]  

(10)

Initial conditions:

\[
f_{\vec{k}}(v_\parallel, 0) = \sum_i w_{i0} \exp[-ik \cdot \vec{x}_{i0}] \delta(v_\parallel - v_\parallel_{i0})
\]  

(11)
Dupree-Tetreault renormalization

Assuming the Fourier modes are statistically independent, the nonlinearity can be approximated by its coherent part:

\[ \mathcal{N}_{\vec{k},\omega} [\phi, f] \simeq \beta_{\vec{k},\omega} \phi_{\vec{k},\omega} - d_{\vec{k},\omega} f_{\vec{k},\omega} \]  

(12)

where

\[ \beta_{\vec{k},\omega} = \frac{1}{B^2} \sum_{\vec{k}',\omega'} \frac{(\vec{k} \cdot \vec{k}' \times \hat{z})^2}{[i(\omega - \omega') - i(k_z - k'_z)v_\parallel]} \phi_{\vec{k}',\omega'} f^*_{\vec{k}',\omega'} \]  

(13)

and

\[ d_{\vec{k},\omega} = \frac{1}{B^2} \sum_{\vec{k}',\omega'} \frac{(\vec{k} \cdot \vec{k}' \times \hat{z})^2}{[i(\omega - \omega') - i(k_z - k'_z)v_\parallel]} |\phi_{\vec{k}',\omega'}|^2 \]  

(14)

Hence,

\[ f_{\vec{k},\omega} \simeq \frac{(\mathcal{L}_k + \beta_{\vec{k},\omega}) \phi_{\vec{k},\omega} + f_{\vec{k}}(v_\parallel, 0)}{(-i\omega + ik_z v_\parallel + d_{\vec{k},\omega})} \]  

(15)
Nonlinear equation for the potential:

Quasineutrality:

\[
\frac{n_0 e}{T_e} \phi_{\vec{k},\omega} = -\frac{n_0 e^2}{T_i} k^2 \rho^2 \phi_{\vec{k},\omega} + e \int dv_{\parallel} f_{\vec{k},\omega}
\]  

(16)

Using the expression for \( f_{\vec{k},\omega} \), this becomes

\[
\frac{n_0 e}{T_e} \epsilon(\vec{k}, \omega) \phi_{\vec{k},\omega} = i \int dv_{\parallel} \frac{f_{\vec{k}}(v_{\parallel}, 0)}{(\omega - k_z v_{\parallel} + i d_{\vec{k},\omega})}
\]  

(17)

where

\[
\epsilon(\vec{k}, \omega) = 1 + k^2 \rho^2 - i \frac{T_e}{n_o} \int dv_{\parallel} \frac{\mathcal{L}_{\vec{k}} + \beta_{\vec{k},\omega}}{(\omega - k_z v_{\parallel} + i d_{\vec{k},\omega})}
\]  

(18)

is the renormalized dielectric. The total damping is \( \gamma_{\vec{k}} = -\epsilon I \frac{\partial \epsilon R}{\partial \omega_{\vec{k}}} \), which will involve \( \beta_{\vec{k},\omega} \) as well as \( d_{\vec{k},\omega} (\approx k^2 D) \).
Drift waves

Assuming $k_z v_{Ti} / \omega \ll 1$,

$$
\epsilon(\vec{k}, \omega) = 1 + k_\perp^2 \rho_s^2 - \frac{\omega_e \gamma_L}{\omega + i\gamma_L} - \frac{k_z^2 c_s^2}{\omega^2} \\
- \left( 1 - \frac{\omega_e}{\omega} - \frac{2k_z^2 c_s^2}{\omega^2} \right) \frac{id_{\vec{k},\omega}}{\omega}
$$

$$
= \epsilon_R(\vec{k}, \omega) + i\epsilon_I(\vec{k}, \omega)
$$

(19)

(20)

(21)

where $\gamma_L$ is the linear damping. Using the linear dispersion relation

$$
1 + k_\perp^2 \rho_s^2 - \frac{\omega_e}{\omega} - k_z^2 c_s^2 / \omega^2 = 0
$$

the imaginary part of the dielectric is

$$
\epsilon_I(\vec{k}, \omega^-) = \frac{\omega_e \gamma_L}{\omega^2 \vec{k}} + \left( k_\perp^2 \rho_s^2 + \frac{k_z^2 c_s^2}{\omega^2 \vec{k}} \right) \frac{d_{\vec{k},\omega}}{\omega^-}
$$

(22)

The total damping is $\gamma_{\vec{k}} = -\frac{\omega^2}{\omega_e} \epsilon_I$. Because the $\beta_{\vec{k},\omega}$ term is included, there is a near cancellation, leading to the small factor multiplying $d_{\vec{k},\omega}$. 
Potential fluctuation spectrum

Using

$$\langle \phi_{\vec{k}\omega} \phi_{\vec{k'}\omega'} \rangle = (2\pi)^4 \langle \phi^2 \rangle_{\vec{k}\omega} \delta(\vec{k} + \vec{k'})\delta(\omega + \omega')$$  \hspace{1cm} (23)$$

and

$$\langle f_{\vec{k}}(v_\parallel, 0) f_{\vec{k}'}^*(v'_\parallel, 0) \rangle = \sum_i w_{i0}^2 F(v_\parallel) \delta(v_\parallel - v'_\parallel)$$  \hspace{1cm} (24)$$

we find

$$\langle \phi^2 \rangle_{\vec{k}\omega} = \text{const} \sum_i w_{i0}^2 \int dv_\parallel F(v_\parallel) \delta(\omega - k_z v_\parallel)$$  \hspace{1cm} (25)$$

The potential fluctuation spectrum is given as the renormalized dielectric response to discrete particle noise.
Conclusions

• The linear form for the fluctuation spectrum becomes invalid even when the plasma is stable, but close to marginal stability.

• Therefore, the Fluctuation-Dissipation Theorem cannot be expected to have the linear form for linearly unstable plasmas.

• Analytical models of noise damping effects require some care – simple $k^2 D$ type estimates are invalid.

• Zonal Flows driven by noise and turbulence require some care in calculating, using knowledge gained from simpler examples.

• ZF potential should be renormalized polarization response to noise and incoherent mode-coupling from drift waves