ROBUST CONTROL OF RESISTIVE WALL MODES IN TOKAMAK PLASMAS

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Mechanical Engineering and Mechanics

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\[ K(s) \]
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Introduction

• Resistive Wall Mode (RWM) is a plasma kink instability whose growth rate is moderated by the influence of a resistive wall.

• FAR-TECH DIII-D/RWM model

• Plasma surface modeled as a toroidal current sheet and the wall modeled with an eigenmode approach.

• State-space model of the plant, whose states are the surrounding wall current and the external control coil currents.
Introduction

- 22 poloidal field probes and saddle loops.
- 12 in-vessel coils are used to oppose the deformation.
- State space model is parameterized with a scalar coupling coefficient $c_{pp}$, which is directly related to the growth rate $\gamma$.

![Relationship between Growth Rate $\gamma$ and $c_{pp}$](image)

- 12 input, 22 output reduced to 3 input, 2 output using a typical quartet configuration for the I-coils and matched filter on the field probes.
The state matrices are given below, where each matrix has a physical correlation to a parameter in DIII-D which is well known, except for the uncertain parameter $c_{pp}$.

\[ \dot{x} = Ax + Bu \]
\[ A = \left( M_{ss} - M_{sp} c_{pp} M_{ps} \right)^{-1} R_{ss} \]
\[ B = \left( M_{ss} - M_{sp} c_{pp} M_{ps} \right)^{-1} \]
\[ y = Cx \]
\[ C = C_{ss} - C_{yp} c_{pp} M_{ps} \]

The transfer function representation of the state matrices

\[ \frac{Y(s)}{U(s)} = G(s) = D + C(sI - A)^{-1} B = G(s, c_{pp}) \]

How does the dynamic response change as $c_{pp}$ changes?
We are interested in designing one controller that stabilizes all the systems → ROBUST CONTROL
Design Goal

- To design a model-based controller that stabilizes the system and meets performance criteria over a large range of growth rate values.

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- Physical growth rate \((\gamma)\) range: 10-5,000 rad/sec
- Corresponding \(c_{pp}\) range: 0.3325-71
- The growth rate relationship through \(c_{pp}\) is treated as an uncertain parameter that acts as a disturbance to a nominal system.
- Robust control tools are applied to the model to stabilize the system over the design range.
- Results in a single controller that ensures stability and performance within the desired growth rate range.
Controller Design

The goal is to design a feedback controller $K$ that robustly stabilize the system for the applicable range of $\Delta$. Defining

$$N = F_l(P, K) \quad \mu(N_{11}) = \frac{1}{\min \{k_m | \det(I - k_m N_{11} \Delta) = 0\}}$$

where $\mu$ is the structured singular value and the term $N_{11}$ isolates the uncertainty from the input and output of the system.

Assuming $N$ and $\Delta$ are stable robust, stability is given by

$$\mu(N_{11}(j\omega)) < 1, \forall \omega$$

and robust performance is given by

$$\mu(N(j\omega)) < 1, \forall \omega$$
System Space Model

The state matrices are given below, where each matrix has a physical correlation to a parameter in DIII-D which is well known, except for the uncertain parameter $c_{pp}$.

$$\dot{x} = Ax + Bu$$

$$A = \left( M_{ss} - M_{sp} c_{pp} M_{ps} \right)^{-1} R_{ss}$$

$$B = \left( M_{ss} - M_{sp} c_{pp} M_{ps} \right)^{-1}$$

Using the Sherman-Morrison formula

$$\left( A_T - b_T C_T D_T \right)^{-1} = A_T^{-1} + \frac{b_T \left( A_T^{-1} C_T \right) \left( D_T A_T^{-1} \right)}{1 - b_T D_T A_T^{-1} C_T}$$

The state matrices can be rewritten as

$$A = A_0 + \sum_{i=1}^{4} \alpha_i A_i$$

$$B = B_0 + \sum_{i=1}^{4} \alpha_i B_i$$

$$C = C_0 + \alpha_5 C_5$$

where each $\alpha$ term is a nonlinear function of the uncertain parameter $c_{pp}$ and every other term is constant.
Linear Fractional Transformation

Using the linear fractional transformation, which is defined by the upper and lower transform for a matrix $M$ as:

$$F_u(M, \Delta_u) = M_{22} + M_{21} \Delta_u (I - M_{11} \Delta_u)^{-1} \ M_{12}$$

The transfer function representation of the state matrices

$$G(s) = D + C(sI - A)^{-1} B$$

can be written as

$$G(s) = F_u\left(M_\alpha, \frac{1}{s}I\right) = M_{a22} + M_{a21} \frac{1}{s} (I - M_{a11} \frac{1}{s})^{-1} \ M_{a12} = M_{a22} + M_{a21} (sI - M_{a11})^{-1} M_{a12}$$

where

$$M_\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_0 + \sum_{i=1}^{4} \alpha_i A_i & B_0 + \sum_{i=1}^{4} \alpha_i B_i \\ C_0 + \alpha_5 C_5 & 0 \end{bmatrix}$$
Model Progression
Singular Value Decomposition

The uncertainty $\alpha$ can be formulated into a linear fractional transform by achieving the smallest possible repeated blocks.

$$J_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = U_i \Sigma_i V_i^* = (U_i \sqrt{\Sigma_i}) (\sqrt{\Sigma_i} V_i^*) = \begin{bmatrix} L_i & R_i \\ W_i & Z_i \end{bmatrix}$$

Denoting $q_i$ as the rank of each matrix $J_i$, introducing the uncertainty

$$\alpha_i J_i = \begin{bmatrix} L_i \\ W_i \end{bmatrix} [\alpha_i I_{q_i} \begin{bmatrix} R_i \\ Z_i \end{bmatrix}]$$

$$M_{\alpha} = M_{11} + M_{12} \alpha_p M_{21} = F_l(M, \alpha_p)$$

$$M_{11} = \begin{bmatrix} A_0 & B_0 \\ C_0 & 0 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} L_1 & \cdots & L_5 \\ W_1 & \cdots & W_5 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} R_1^* & Z_1^* \\ \vdots & \vdots \\ R_5^* & Z_5^* \end{bmatrix}$$

$$\alpha_p = \begin{bmatrix} \alpha_1 I_{q_1} & 0 \\ \vdots & \vdots \\ 0 & \alpha_5 I_{q_5} \end{bmatrix}$$

$$G(s) = F_u \left( M_{\alpha}, \frac{1}{s} \right) = F_u \left( F_l(M_{\alpha}, \alpha_p), \frac{1}{s} \right)$$
The uncertainty $\alpha_p$ is still a nonlinear matrix based on the uncertain parameter $c_{pp}$. This parameter is extracted using by “pulling out the $\delta$”, where $\delta$ is the normalized $c_{pp}$.

The final result is a linear fractional transform

$$\alpha_1 = \frac{d + \delta e}{1 - a(d + \delta e)}$$

$$Q_1 = \begin{bmatrix} d & e \left(1 + \frac{ad}{1 - ad}\right) \\ 1 - ad & 1 - ad \\ 1 - ad & ae \\ 1 - ad & 1 - ad \end{bmatrix}$$

$$\alpha_1 = F_l(Q_1, \delta)$$

$$\Delta = \delta I \quad |\delta| \leq 1$$
Model Progression
Pulling out the “$\delta$”

Finally this can be substituted into the plant and simplified using several properties of the linear fractional transform

$$G(s) = F_u \left( M_\alpha, \frac{1}{s} I \right) = F_u \left( F_l \left( M_\alpha, \alpha_p \right), \frac{1}{s} I \right) = F_u \left( F_l \left( M_\alpha, F_l(Q, \Delta) \right), \frac{1}{s} I \right)$$

$$G(s) = F_u \left( F_l \left( R, \Delta \right), \frac{1}{s} I \right) = F_l \left( F_u \left( R, \frac{1}{s} I \right), \Delta \right) = F_l \left( P', \Delta \right) = F_u \left( P, \Delta \right)$$

where

$$R = \begin{bmatrix} M_{11} + M_{12}Q_{11}M_{21} & M_{12}Q_{12} \\ Q_{21}M_{21} & Q_{22} \end{bmatrix} \quad P' = F_u \left( R, \frac{1}{s} I \right) = \begin{bmatrix} P'_{11} & P'_{12} \\ P'_{21} & P'_{22} \end{bmatrix}$$
Model Progression

\[ \frac{1}{s} I \]

\[ M_\alpha \]

\[ \alpha_p \]

\[ \frac{1}{s} \]

\[ M \]

\[ Q \]

\[ \Delta \]

\[ \frac{1}{s} I \]

\[ P' \]

\[ P \]

\[ \Delta \]

\[ \Delta \]

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D-K Iteration

- No direct method to synthesize a \( \mu \)-optimal controller.

- DK-iteration combines \( H_\infty \) synthesis and \( \mu \)-analysis.

- This method starts with the upper bound on \( \mu \) in terms of the scaled singular value

\[
\mu(N) \leq \min_{D \in \Phi} \sigma(DND^{-1})
\]

where \( \Phi \) is the set of matrices \( D \) which commute with \( \Delta \), i.e., \( D\Delta = \Delta D \).

- Then, the controller that minimizes the peak value over frequency of this upper bound is found, namely

\[
\min_K \left( \min_{D \in \Phi} ||DN(K)D^{-1}||_\infty \right)
\]
The controller is designed by alternating between the two minimization problems until reasonable performance is achieved.

Follow the steps until $\|DN(K)D^{-1}\|_\infty < 1$, or $H_\infty$ norm doesn’t decrease.

1. **K-step.** Design an $H_\infty$ controller for the scaled problem with fixed $D(s)$.

2. **D-step.** Find $D(j\omega)$ to minimize upper bound at each frequency with fixed $N$.

3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum-phase transfer function $D(s)$ and go to step 1.
Controller Simulation Results

- High rotating plasma, the growth rate $\gamma$ ranges from 10 $rad/s$ to 5,000 $rad/s$. This results in a range for the uncertain parameter $c_{pp}$ that goes from 71 to 0.3325.

- Controllers designed with a smaller, more unstable nominal $c_{pp}$ value produce the widest range of stability for $c_{pp}$.

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<th>Performance Ranges</th>
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<td>DK</td>
<td>0 - 7,437 rad/s</td>
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</tr>
<tr>
<td>DK</td>
<td>0 - 7,437 rad/s</td>
<td>0 - 6,459 rad/s</td>
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Performance Constraints                      Stability\Performance Ranges
Initial Condition Response

RWM Mode Amplitude Time Response for $\gamma = 10$ rad/sec

RWM Mode Amplitude Time Response for $\gamma = 5,000$ rad/sec

DK Controller
NCF Controller
Initial Condition Response

First Coil Voltage Applied Time Response for $\gamma = 10 \text{ rad/sec}$

First Coil Voltage Applied Time Response for $\gamma = 5,000 \text{ rad/sec}$
Step Response

RWM Mode Amplitude Time Response for $\gamma = 10$ rad/sec

RWM Mode Amplitude Time Response for $\gamma = 5,000$ rad/sec
Step Response

First Coil Voltage Applied Time Response for $\gamma = 10$ rad/sec

First Coil Voltage Applied Time Response for $\gamma = 5,000$ rad/sec

- DK Controller
- NCF Controller

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Time-varying Growth Rate Response

\( \gamma \) and \( c_{pp} \) Trajectories versus Time

\( \gamma \) Growth Rate (rad/sec)

Time (s)

RWM Mode Amplitude Time Response for Step \( \gamma \)

Dashed line: DK Controller
Solid line: NCF Controller

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Time-varying Growth Rate Response

Voltage Applied Time Response

- Voltage Applied (V)
- Time (s)

DK Controller
NCF Controller
Sinusoidal Growth Rate Response

\[
\gamma \text{ and } c_{pp} \text{ Trajectories versus Time}
\]

\[
\text{RWM Mode Amplitude Time Response for Sinusoidal } \gamma
\]

\[
\text{DK Controller}
\]

\[
\text{NCF Controller}
\]
Sinusoidal Growth Rate Response

Voltage Applied Time Response

- Voltage Applied (V)
- Time (s)

DK Controller
NCF Controller

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Conclusions

- Toroidal current sheet model for the DIII-D tokamak plasma was restructured into a robust control framework, isolating the RWM growth rate $\gamma (c_{pp})$, the key term of RWM instability.

- With the system model in this framework, the parametric $DK$-iteration method was applied to develop a structured-singular-value based robust controller for a pre-determined range of $\gamma$.

- Since the plasma RWM growth rate can vary throughout the operation of the DIII-D tokamak, the design of a controller that can stabilize the system over the entire physical range of $\gamma$ is critical.

- In terms of robust stability, this method eliminates the need of online identification and controller scheduling.