STRATEGY FOR ERROR FIELD CONTORL

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November 20, 2011

Objectives:

1. Compensate for uncertainties not only in tokamak construction but also in plasma physics.

2. Validate a strategy applicable to future fusion systems.

3. Determine accuracy with which future tokamaks must be built.

*Basic ideas in Fusion Science and Technology **59**, 561 (2011).*

Two Central Concepts

which had no role in earlier ITER error-field studies:
On Error Field Correction in Fusion Reactors, Formisano & Martone,
IEEE Trans. Appl. Supercond. **20**, 547 (2010).

1. If $\vec{B}_x(\vec{x})$ is the magnetic field due to currents outside a region bounded by a control surface, then

   $\vec{B}_x(\vec{x})$ is uniquely specified by $\vec{B}_x \cdot \hat{n}$ on the surface and the total enclosed toroidal flux produced by these currents.

   $\vec{B}_x \cdot \hat{n}$ described by a set of fluxes

$$\Phi_i^{(x)} \equiv \oint f_i(\theta,\varphi)\vec{B}_x \cdot d\vec{a}, \text{ so } \vec{B}_x \cdot \hat{n} = w\sum \Phi_j^{(x)} f_j(\theta,\varphi)$$

$$\oint f_i f_j w\, da = \delta_{ij}, \quad w>0, \quad \text{and} \quad \oint w\, da = 1.$$

f_i*'s* Fourier functions with choice $wda = d\theta d\varphi$.

2. Plasma has wide range of sensitivities to different $\vec{B}_x \cdot \hat{n}$'s.

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σ_c is a critical magnetic flux: $\sigma_c^2 = \oint \left(\frac{\vec{B}_x \cdot \hat{n}}{w} \right)^2 w da.$

Plasma has greatest sensitivity to external distribution $\left(\vec{B}_x \cdot \hat{n}\right)_1$ that has an unacceptable effect with the smallest $\sigma_c=\sigma_1$.

Plasma has second greatest sensitivity to external distribution $\left(\vec{B}_x \cdot \hat{n}\right)_2$ that is orthogonal $\left(\vec{B}_x \cdot \hat{n}\right)_1$ but otherwise minimizes σ_c.

$$\oint \left(\vec{B}_x \cdot \hat{n}\right)_1 \left(\vec{B}_x \cdot \hat{n}\right)_2 \frac{da}{w} = 0.$$

Each $\left(\vec{B}_x \cdot \hat{n}\right)_i$ in a sensitivity series is unacceptable when

$$\left(\vec{B}_x \cdot \hat{n}\right)_i = w\sigma_i F_i(\theta,\phi), \text{ where } \oint F_i F_j w\, da = \delta_{ij}.$$

$1/\sigma_i^2$ eigenvalues and $F_i(\theta,\varphi)$ eigenfuctions of sensitivity matrix

Two Central Concepts (Summary)

1. $\vec{B}_x(\vec{x})$, the magnetic field in a region bounded by a control surface due to currents outside, is uniquely defined by $\vec{B}_x \cdot \hat{n}$ on the surface *and the total enclosed toroidal flux produced by these currents.*

$$\Phi_i^{(x)} \equiv \oint f_i(\theta,\varphi)\vec{B}_x \cdot d\vec{a}, \text{ so } \vec{B}_x \cdot \hat{n} = w\sum \Phi_j^{(x)} f_j(\theta,\varphi).$$

2. Plasma has a wide range of sensitivities to different $\vec{B}_x \cdot \hat{n}$'s.

Sensitivity matrix $S_{ij} \equiv \sum_k \left(\oint f_i F_k w\, da\right) \frac{1}{\sigma_k^2} \left(\oint F_k f_j w\, da\right).$

Eigenvalues $1/\sigma_k^2$ range over orders of magnitude.

Error fields are controlled if and only if the external field $\vec{B}_x \cdot \hat{n}$ on a control surface between the coils and the plasma can be modified to ensure $\vec{\Phi}_x^\dagger \cdot \vec{S} \cdot \vec{\Phi}_x < 1$.

Properties of Plasma Sensitivity

Eigenvalues $1/\sigma_i^2$ and eigenfuctions $F_i(\theta,\varphi)$ of the sensitivity matrix $\overleftrightarrow{S}$ can be determined theoretically or empirically.

First distribution, $\left(\vec{B}_x \cdot \hat{n}\right)_1 \propto F_1(\theta,\varphi)$, generally drives least stable plasma kink.

Properties of higher $\left(\vec{B}_x \cdot \hat{n}\right)_i$'s far less studied, though assumptions are required for error field control to be feasible.

Feasibility highly dependent on increase of σ_i with *i*.

A slow increase:

Sudden switch from no to many $\left(\vec{B}_x \cdot \hat{n}\right)_i$ requiring control.

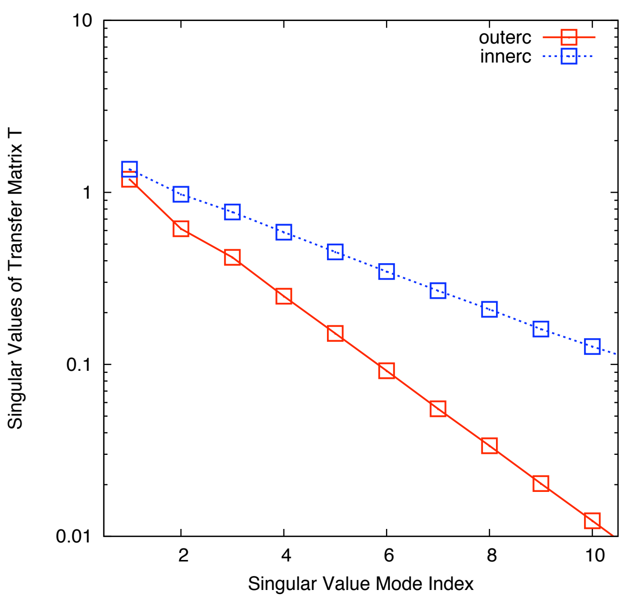
Will discuss applications before typical $\overleftrightarrow{S}$ matrices are considered.

Error Field Control using Coil Currents

On a control surface just outside the plasma $\vec{\Phi}_x^{\dagger} \cdot \overleftrightarrow{S} \cdot \vec{\Phi}_x < 1$ required.

External error field $\vec{\Phi}_x = \overleftrightarrow{T} \cdot \vec{\Phi}_c + \overleftrightarrow{M} \cdot \vec{J}$ where $\vec{B}_x \cdot \hat{n} = w \sum \Phi_j^{(x)} f_j(\theta,\varphi)$.

- $\vec{\Phi}_c$ gives error field due to main coils on a control surface just on the plasma side of the coils.
- $\vec{J}$ components give currents in the control coils
- $\overleftrightarrow{T}$ is the transfer matrix from the coil to the plasma control surf.



Singular values of $\overleftrightarrow{T}$:

Red: coils to plasma control surface.

Blue: first wall to plasma control surface.

Rapid decay of singular values eases error control.

Note log scale for SVD singular values.

Singular Value Decomposition and the Pseudo-Inverse

Essentially any matrix can be represented as $\overleftrightarrow{M} = \overleftrightarrow{U} \cdot \overleftrightarrow{m} \cdot \overleftrightarrow{V}^{\dagger}$

$\overleftrightarrow{U}$ and $\overleftrightarrow{V}$ are unitary, $\overleftrightarrow{U}^{\dagger} \cdot \overleftrightarrow{U} = \overleftrightarrow{1}$, and $m_{ij} = m_i \delta_{ij}$ with $m_i \geq 0$ real.

Inverse $\overleftrightarrow{M}^{-1} = \overleftrightarrow{V} \cdot \overleftrightarrow{m}^{-1} \cdot \overleftrightarrow{U}^{\dagger}$ where $\left(\overleftrightarrow{m}^{-1}\right)_{ij} = \frac{1}{m_i} \delta_{ij}$.

Pseudo-inverse $\overleftrightarrow{M}_k^{-1}$ retains only the k largest m_i in forming inverse.

If the singular values m_i are arranged in order (largest to smallest) the condition number $C_k \equiv \frac{m_1}{m_k} \geq 1$.

Maximum obtainable accuracy when

$$\frac{1}{C_k} \sim \text{fractional uncertainty.}$$

Optimal Control Coil Currents

Minimize $\vec{\Phi}_x^\dagger \cdot \overleftrightarrow{S} \cdot \vec{\Phi}_x$ where $\vec{\Phi}_x = \overleftrightarrow{T} \cdot \vec{\Phi}_c + \overleftrightarrow{M} \cdot \vec{J}$

Answer is $\vec{J} = -\overleftrightarrow{\mathcal{C}}_k \cdot \vec{\Phi}_c$ where $\overleftrightarrow{\mathcal{C}}_k \equiv \left(\overleftrightarrow{M}^\dagger \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{M}\right)_k^{-1} \cdot \overleftrightarrow{M}^\dagger \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{T}$.

$\left(\overleftrightarrow{M}^\dagger \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{M}\right)_k^{-1}$ means a pseudo-inverse using k singular values.

Let $\overleftrightarrow{\mathcal{T}}_k \equiv \overleftrightarrow{T} - \overleftrightarrow{M} \cdot \overleftrightarrow{\mathcal{C}}_k = \left\{\overleftrightarrow{1} - \overleftrightarrow{M} \cdot \left(\overleftrightarrow{M}^\dagger \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{M}\right)_k^{-1} \cdot \overleftrightarrow{M}^\dagger \cdot \overleftrightarrow{S}\right\} \cdot \overleftrightarrow{T}$ and $\overleftrightarrow{\mathcal{R}}_k \equiv \overleftrightarrow{\mathcal{T}}_k^\dagger \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{\mathcal{T}}_k$.

Then, $\vec{\Phi}_x^\dagger \cdot \overleftrightarrow{S} \cdot \vec{\Phi}_x = \vec{\Phi}_c^\dagger \cdot \overleftrightarrow{\mathcal{R}}_k \cdot \vec{\Phi}_c$.

1. Largest eigenvalue of $\overleftrightarrow{\mathcal{R}}_k$ defines residual error field sensitivity.
2. The condition number of $\left(\overleftrightarrow{M}^\dagger \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{M}\right)_k^{-1}$ determines sensitivity to uncertainties. Retaining too large a k gives intolerable errors.
3. k is the effective number of independent control coils.

TWO STRATEGIES FOR ERROR FIELD CONTROL

1. Dynamic Error Field Control

   Sense plasma response to errors and respond using feedback.

2. Preprogrammed Error Field Control

   Accumulate knowledge of the error fields associated with the equilibrium coils. Use the known currents in these coils to adjust the currents in the correction coils.

Error field control often thought of as Dynamic Error Field Control. Works when a single external field distribution requires control.

If several external distributions require control, Preprogrammed Error Field Control appears better—particularly in future fusion systems with limited diagnostics.

Constraints on error field control separate the two strategies.

Constraints on Error Field Control

1. Only a few, *k*, external magnetic perturbations can be controlled. Control doesn't mean error elimination or even reduction.

2. External plasma response $\delta\vec{B}_p\cdot\hat{n} = w\sum\Phi_i^{(p)} f_i(\theta,\varphi)$ need not have the same spatial distribution as the external perturbation.

   For small perturbations, $\vec{\Phi}_p = \left(\overleftrightarrow{P} - \overleftrightarrow{1}\right)\cdot\vec{\Phi}_x$ but:

   (a) Even in ideal MHD, $\overleftrightarrow{P}$ is non-Hermitian so driving perturbation $\vec{\Phi}_x$ and plasma response $\vec{\Phi}_p$ are not parallel.

   (b) When perturbation applies a toroidal torque to plasma, $\vec{\Phi}_x$ and $\vec{\Phi}_p$ must have a toroidal phase shift.

3. The external perturbations that require correction can depend on the plasma equilibrium—certainly the plasma response does.

4. The $\vec{\Phi}_x$ to which the plasma is most sensitive generally produces a large plasma response—error field amplification—but error fields can produce an unacceptable Neoclassical Toroidal Viscosity (NTV) with only a weak external plasma response.

5. Error control coils should not be significantly further from the plasma than the source of the error, but can be a comparable distance. Current need change only on equilibrium time scale.

Control coils for producing beneficial effects—such as ELM control—may need to be very close to plasma. RWM control coils must have a faster response.

Areas of Important Research

1. Convergence properties of external field errors $\left(\overleftrightarrow{W} \text{ matrix}\right)$.

   Actual source of error—which coils are out of position—is unimportant unless coils are to be repositioned.

   If different error fields have different expected amplitudes, the error field analysis should include a weight matrix $\overleftrightarrow{T} \rightarrow \overleftrightarrow{T} \cdot \overleftrightarrow{W}$.

   Then, $Trace\left[\mathcal{R}_k\right] < 1$ required for acceptable control.

Weight matrix $\overleftrightarrow{W}$: *(Not yet studied numerically or in as-built machines.)*

In r^{th} Monte Carlo realization of the coil system, suppose $\Phi_i^{(c)} = F_{ri}$.

$$\text{Let } \mathcal{F}_{ij} \equiv \frac{1}{r_{tot}} \sum_{r=1}^{r_{tot}} F_{ri} F_{rj}, \text{ so } \overleftrightarrow{\mathcal{F}} = \overleftrightarrow{U} \cdot \overleftrightarrow{\mathcal{W}}^2 \cdot \overleftrightarrow{U}^\dagger, \text{ where } \mathcal{W}_{ij} = \mathcal{W}_i \delta_{ij}.$$

Let $\vec{\Phi}_c = \overleftrightarrow{W} \cdot \vec{\Phi}_a$, where $\overleftrightarrow{W} \equiv \overleftrightarrow{U} \cdot \overleftrightarrow{\mathcal{W}} \cdot \overleftrightarrow{U}^\dagger$ then all components of $\vec{\Phi}_a$ have the same RMS expected amplitude--unity.

2. Properties of sensitivity matrix $\overleftrightarrow{S}$

a. Two types of properties

i. a simple property

Opening of an n=1 island at the q=2 surface, which is controlled by a single external magnetic field distribution.

ii. a compound property

Neoclassical enhancement of rotation damping due to toroidal variation in the magnetic field strength depends in principle on an arbitrarily large number of external field distributions.

b. Dependence of $\overleftrightarrow{S}$ on plasma equilibrium

The spatial distribution of the error fields that must be controlled and the required accuracy depend on the plasma equilibrium.

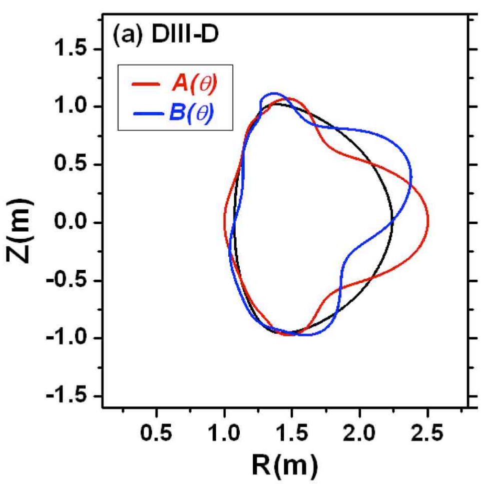
Backup Slides on Types of Plasma Properties

Simple Plasma Property

Drive for an n=1 island at the q=2 surface is archetypal for a simple plasma property.

Measured by jump $\Delta_{mn} \equiv \frac{d}{d\psi}\left(\frac{\delta\vec{B}\cdot\vec{\nabla}\psi}{\vec{B}_0\cdot\nabla\varphi}\right)_{mn}$ across $q=\frac{m}{n}=2$ surface.

A code like IPEC can find the $1\times\infty$ matrix between Δ_{21} and an arbitrary external perturbation on the plasma surface $\vec{\Phi}_x$. A $1\times\infty$ matrix has only one singular value and associated eigenfunction $\vec{B}_x\cdot\hat{n}$.



Black contour is plasma boundary

Colored contours give the unique $\delta\vec{B}_x\cdot\hat{n}$ that drives the q=2 island,

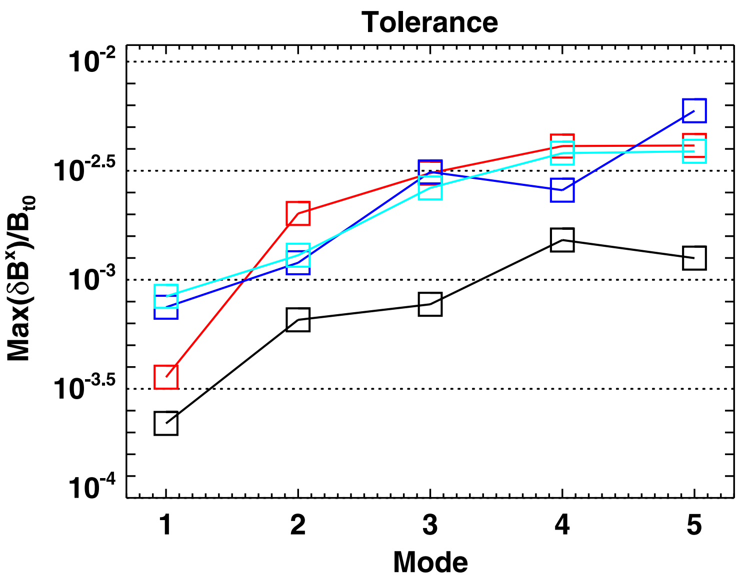
$$f_1 = A(\theta)\cos\varphi + B(\theta)\sin\varphi .$$

Park et al, PRL <u>**99**</u>, 195003 (2007).

Simple Plasma Property in Multiple Equilibria

Definition: an *envelope control surface* is outside of every plasma equilibrium in a set but everywhere close to at least one equilibrium.

Only a moderate number of normal fields on the envelope control surface is required to eliminate a significant drive for an island on any rational surface of any ITER equilibrium.



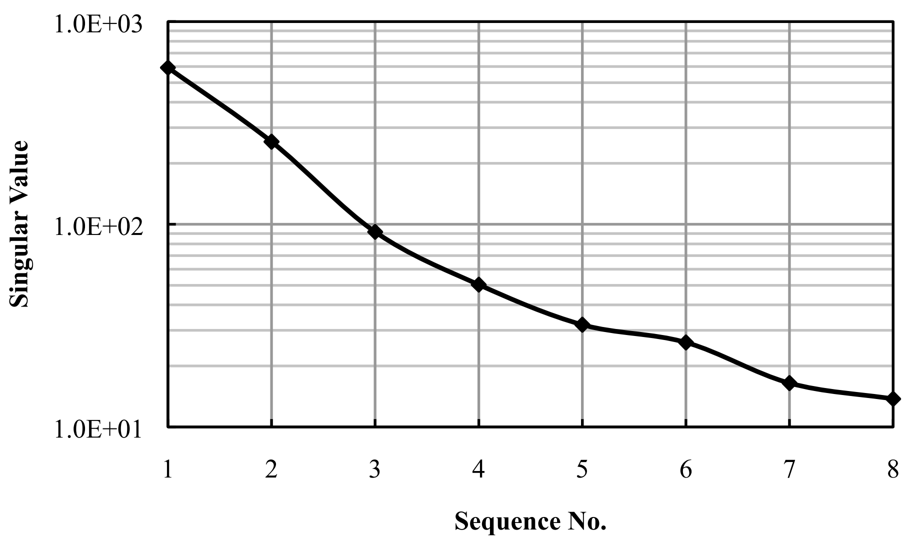
Island drive in ITER, $\propto 1/\sigma_i$, for n = 1 (black), n = 2 (red), n = 3 (blue) and n = 4 (light blue). For all rational surfaces *q=m/n* and for three ITER equilibria *(inductive, hybrid, advanced).*

Park et al, N.F. **48**, 045006 (2008).

Compound Plasma Property

The enhanced transport due to the toroidal variation in the magnetic field strength depends in principle on an infinite number of external field distributions.

A measure of the enhancement is the effective magnetic ripple on each magnetic surface ε_{eff}.



The singular values σ_i associated with ε_{eff} go to zero exponentially for the W7-X stellartor.

Boozer & Ku, PoP **18**, 052501 (2011).