On the Relation Between 3-D Equilibrium and Stability

by

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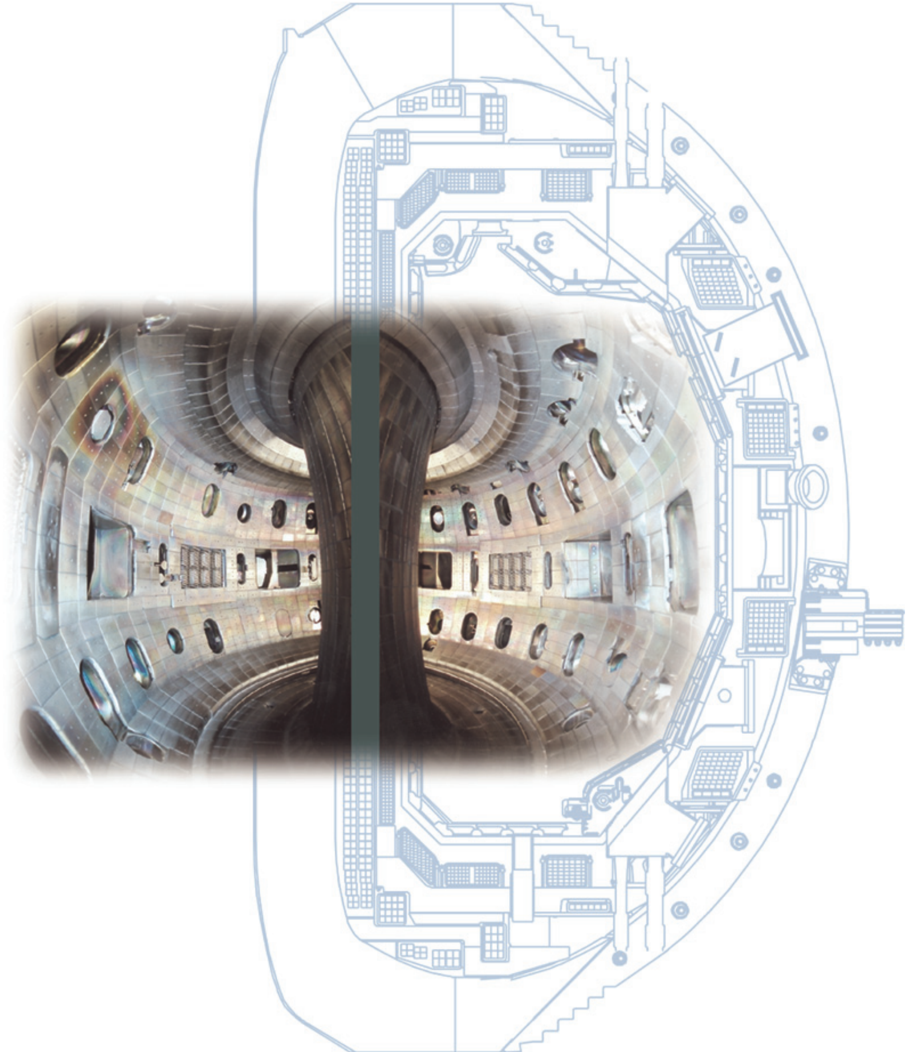

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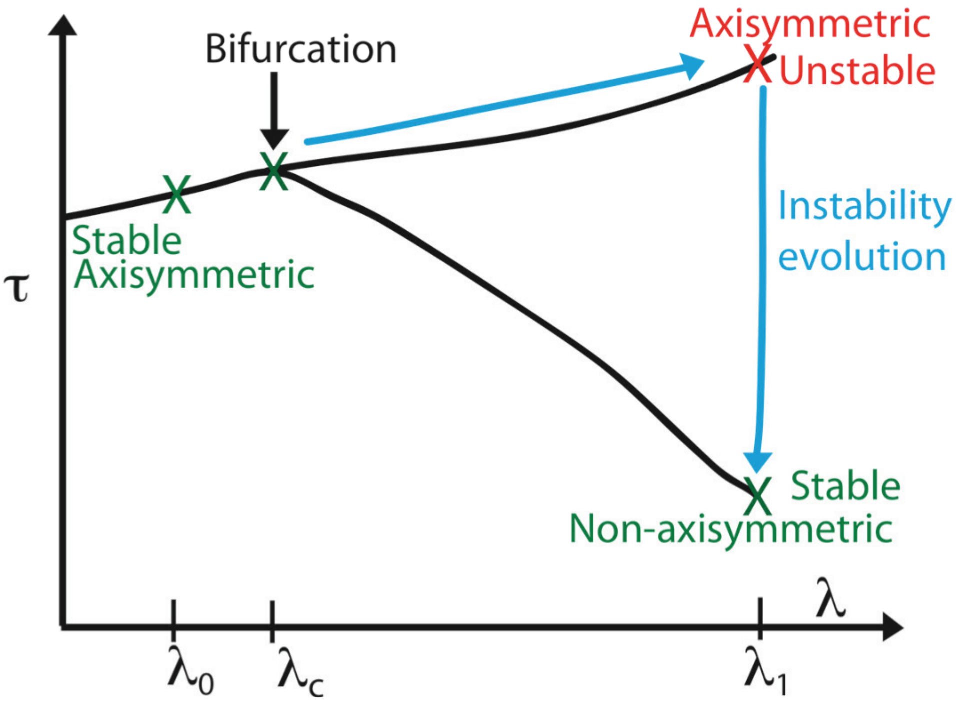
Understanding Plasma Response to Non-Axisymmetric Perturbations is a Vital Area of Fusion Research

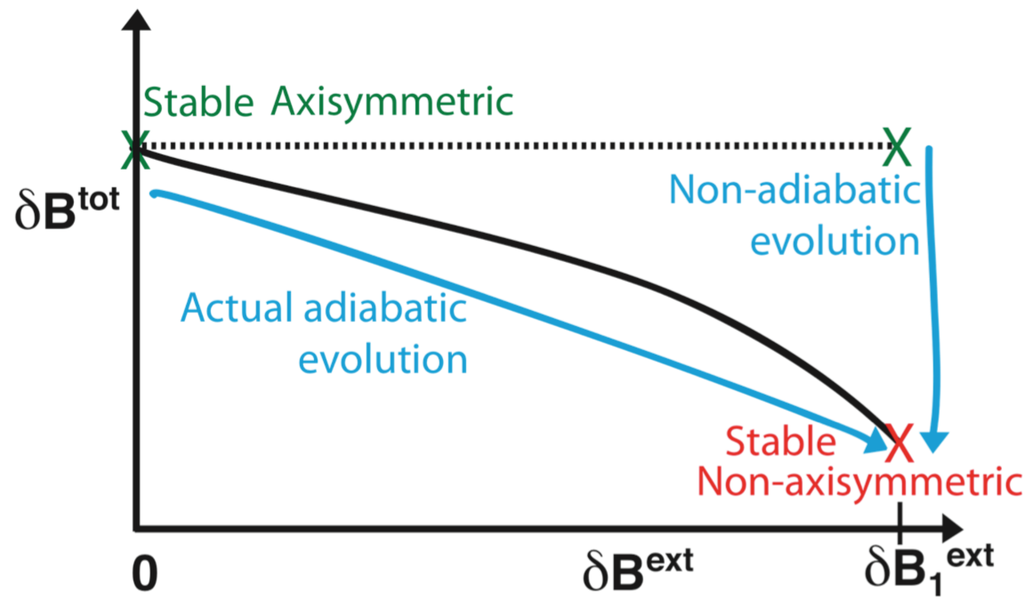
- **3-D perturbations can arise from several sources:**
  - Unintended error fields
  - Imposed external non-axisymmetric coils (intended error fields)
  - Nonlinearly saturated instabilities
- **Plasma response is a key ingredient in determining the consequences in each case:**

  ⇒ Plasma can amplify, suppress or otherwise modify perturbation !
- **Problem of plasma response can be viewed from several viewpoints:**
  - Linear and Nonlinear
  - Perturbed nearby equilibrium or dynamic evolution
- **Each viewpoint informs the others:**
  - None is currently guaranteed to provide the right answer:
    - Linear response is well formulated but not the full story
    - Nearby equilibrium approach does not guarantee right solution
    - Nonlinear response via dynamic evolution is numerically challenging
  - Relation between viewpoints can yield insights

Stability in Presence of Non-axisymmetric Fields can be Considered as a Perturbed Equilibrium Problem

- **Plasma evolves as a parameter is varied:**
- **For unstable 2-D equilibrium:**
  - Bifurcation as parameter λ varies:
  - ⇒ Non-axisymmetric instability
  - ⇒ Evolves to new lower energy 3-D state
- **For external field generated perturbations:**
  - Parameter is external field:
  - ⇒ Transition to a new 3-D equilibrium from a stable 2-D state with increasing non-axisymmetric field

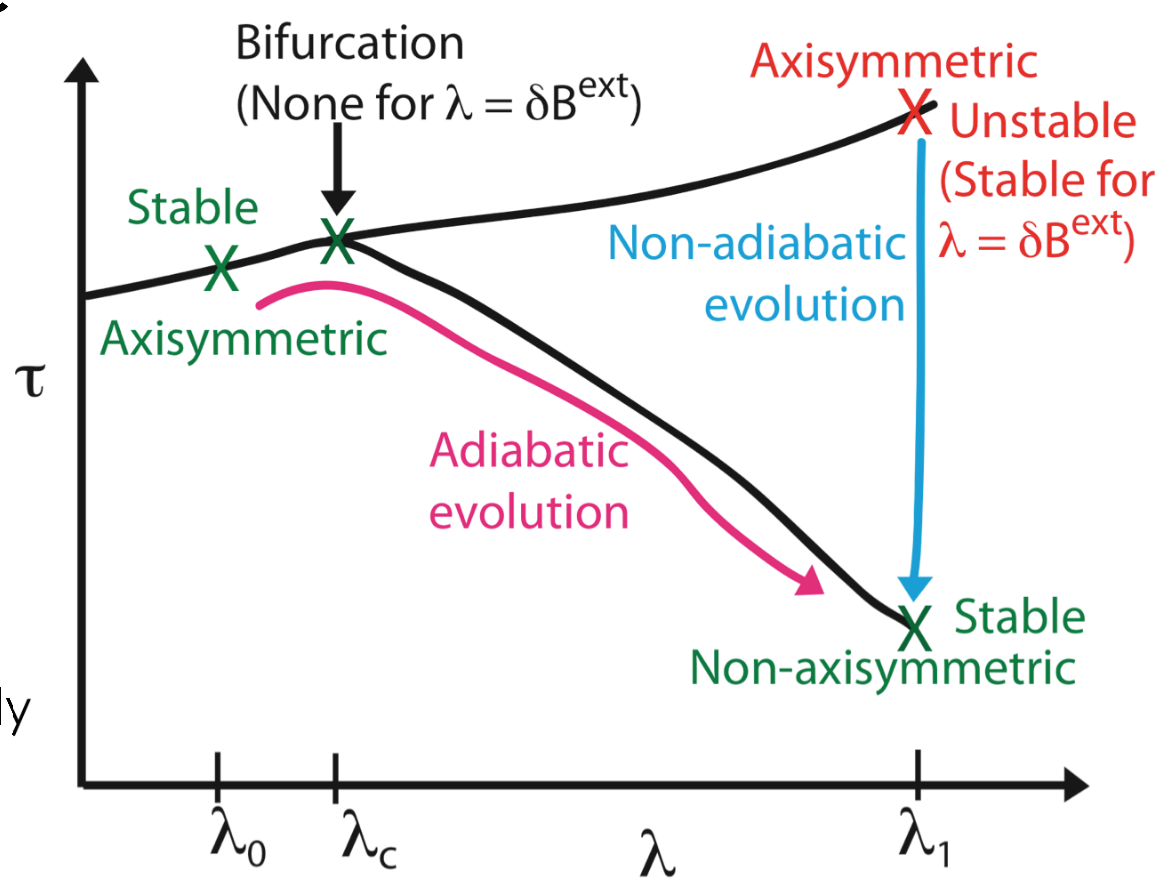




Evolution to New State May Also be Adiabatic or Non-adiabatic

- **Evolution from 2-D equilibrium to 3D equilibrium via instability may be adiabatic or non-adiabatic**
- **External field generated perturbations are typically adiabatic:**
  - Slowly increasing nonaxisymmetric field

  For external field generated perturbations one can still try to find non-axisymmetric equilibria by considering external field turned on rapidly



- **Issue in either case:**
  - Adiabatic evolution
  - Non-adiabatic evolution

is accessibility of nearby nonaxisymmetric equilibria

Two Alternative Viewpoints Can Each be Considered: Either Stability or Perturbed Equilibrium

- **Linear and Nonlinear stability both have counterparts in the stability and the nearby equilibrium formulations**

	Linear:	Nonlinear:
Stability viewpoint:	Forced eigenvalue	3D Extended MHD stability
Perturbed equilibrium viewpoint:	Basis expansion (NMA/IPEC)	Nearby equilibrium (Generalized AIMHD)

Considering all four approaches together leads to better insights into each one

Each approach complements and informs the other

Linear Response can be Studied from Either Stability or Perturbed Equilibrium Viewpoints

Stability

- **Linear case can be formulated naturally as forced eigenvalue problem:**
  - Replace linearized MHD eigenvalue problem: $L\xi - \lambda\xi$
  - By an inhomogeneous forcing function formulation or inhomogeneous boundary conditions that can be reformulated as an inhomogeneous forcing function: $L\xi - F$

Perturbed Equilibrium

- **Expand plasma response in terms of set of basis functions of linearized ideal MHD operator:**

$$\mathcal{L}\xi_i = \lambda_i \xi_i \quad (i=1,N)$$

  - Project this complete set of eigenmodes on to the boundary to yield a complete set of boundary displacements $X_i(\chi,\phi)$ (Nuhrenberg, Boozer Phys Plasmas, 2003)

- SVD fit of coefficients of $X_i(\chi,\phi)$ to imposed boundary perturbation:

$$X_{ext}(\chi,\phi) = \sum_i a_i X_i(\chi,\phi)$$

- Coefficients provide complete internal plasma response:

$$\xi(\chi,\phi) = \sum_i a_i \xi_i(\chi,\phi)$$

Relation Between Linearized Stability and Perturbed Equilibrium Viewpoints is Through Greens Function

- **Inhomogeneous linearized dynamics problem:**

$$\left(\partial^2 \xi \big/ \partial t^2\right) - L\xi = F\, e^{i\omega_0 t} \quad \Rightarrow L\xi + \omega_0^2 \xi = -F$$

- **Formal Greens function solution:** $\xi(r) = \left(L + \omega_0^2\right)^{-1} F\, e^{i\omega_0 t} = \int G(r,r') F(r')\, dr'\, e^{i\omega_0 t}$

- **Forced eigenvalue problem solution and forcing function can also be expanded in eigenvectors of homogeneous linear operator:**

$$\mathfrak{L}\xi_i = \lambda_i \xi_i \quad (i=1,N) \quad \lambda_i \equiv -\omega_i^2 \qquad\qquad \xi(r) = \sum_i a_i \xi_i(r) \qquad F(r) = \sum_i f_i \xi_i(r)$$

$$L\xi + \omega_0^2 \xi = -F \quad \Rightarrow \sum_i a_i \left(\omega_i^2 - \omega_0^2\right) \xi_i(r) = -\sum_i f_i \xi_i(r) = -\sum_i \xi_i(r) \int f(r') \xi_i(r') dr'$$

- **Thus the 'linearized dynamics' solution can be related to the perturbed equilibrium solution through an expression for the Greens Function in terms of the perturbed equilibrium expansion:**

$$\Rightarrow \xi = \int \sum_i \left(\xi_i(r) \xi_i(r') \big/ \left(\omega_i^2 - \omega_0^2\right) \right) f(r') dr'\, e^{i\omega_0 t} \Rightarrow G(r,r') \equiv \sum_i \left(\xi_i(r) \xi_i(r') \big/ \left(\omega_i^2 - \omega_0^2\right) \right)$$

Nonlinear Response Can Be Studied From Either Stability Or Perturbed Equilibrium Viewpoints

Stability

- **Extended MHD stability code can be used to determine final nonlinear saturated state from an initial unstable equilibrium:**
  - Full details of dynamics
  - Time consuming:

Perturbed Equilibria

- **Find a nearby stable nonaxisymmetric equilibrium:**
  - Accessibility of final state from the initial state is key issue

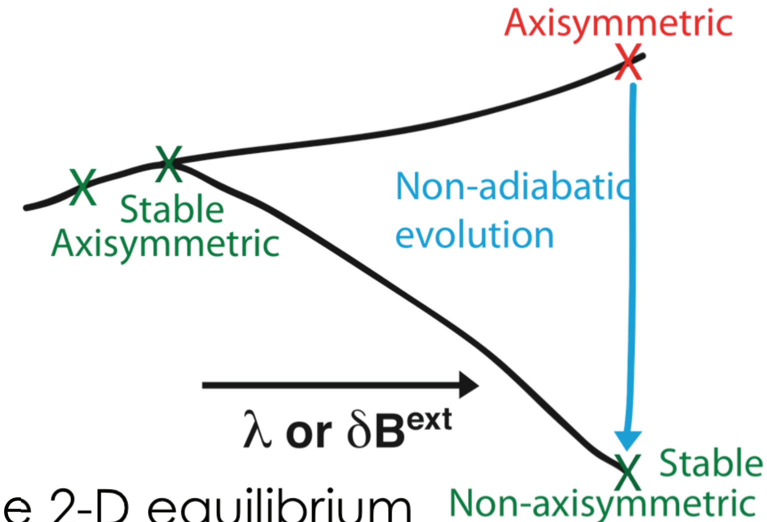
  $\Rightarrow$ Find a set of constraints relating initial unstable axisymmetric state to final stable nonaxisymmetric state

- **Key is to find constraints or invariants relating the unstable 2-D system with a 3-D system**

- **Can the two viewpoints be related as in linear case?**
  - Forcing function formulation is not as convenient in nonlinear case

Nonlinear Perturbed Equilibrium Approach: Find Nearby Accessible Stable Non-axisymmetric Equilibrium

- **Seek a nearby equilibrium from either an:**
  - Unstable 2-D equilibrium or
  - Equilibrium with 3-D external perturbation



Simplest Approach:

- **Add perturbation to base 2D equilibrium:**
  - External field or:
    Computed from linear stability code for base 2-D equilibrium
- **Treat this as an initial guess for the nearby 3-D equilibrium**
- **Solve for 3-D force balance using 3-D equilibrium code:**
  - ⇒ Nearby nonaxisymmetric equilibrium with perturbed boundary shape
  - Inherently nonlinear despite use of linear eigenmodes as 'initial guess'

- **But there is no guarantee that this is the state found dynamically:**
  - External perturbed field case is 'non-adiabatic' path to re-establishing force balance but actual dynamic path is generally adiabatic
  - Instability case points in direction of steepest gradient to lower energy but that is all that can be said

AIMHD (Jensen 1991, 2001): Look For a Set of Invariants Relating 2-D System With Nearby 3-D System

- **For ideal case local helicity is invariant**
  - ⇒ Infinite number of invariants and no topology change
- **For Taylor theory only global helicity survives as a single invariant**
  - ⇒ Force free state is the only possible one

For AIMHD: search instead for a finite number of invariants

- **Ideal case: For any functions $G_n(y)$, n = 1,2,…**

$$K_v = \int G_v(\psi)\, d\sigma \qquad \Rightarrow \qquad \frac{dK_v}{dt} = \int \frac{dG_v}{d\psi}\frac{\partial\psi}{dt} d\sigma \sim \eta H \rightarrow 0$$

- **AIMHD (Jensen 2001):**
  - Select functions $G_n(y)$, n = 1,2,… associated with a finite set of flux surfaces y_n:

$$\frac{dG_v}{d\psi} = \begin{cases} 0 & \psi < \psi_{v-1}\,,\ \psi > \psi_{v+1} \\ (\psi - \psi_{v-1})/(\psi_v - \psi_{v-1}) & \psi_{v-1} < \psi < \psi_v \\ (\psi - \psi_{v+1})/(\psi_v - \psi_{v+1}) & \psi_v < \psi < \psi_{v+1} \end{cases}$$

While this arbitrary choice does not work the general idea of selecting a few constraints more judiciously seems a valid physical approach

Simple Nearby Equilibrium Approach is Really an Example of 'Generalized' Almost Ideal MHD

- **In simple approach AIMHD invariants are "buried" in the numerical details of the equilibrium code:**
  - VMEC imposes fixed topology of nested surfaces but not stellarator symmetry
  - PIES HINT and SIESTA will allow topology breaking but impose stellarator symmetry

These symmetries are effectively invariants imposed on both the 2D and final 3D solutions

- **Resulting 3-D equilibrium also depends on what is held fixed:**
  - Total current ?, Total flux?, β, ?, β_p ?,……
- **Relaxation from initial guess to force balance can be treated as a kind of 'fake dynamics' with iteration step acting like a time step:**
  - For VMEC, PIES, and SIESTA this 'dynamics' is obscure
  - For HINT the relaxation follows the actual dynamical equations but with enhanced inertia to speed up convergence

$\Rightarrow$ **The numerically imposed constraints are not necessarily those respected by the actual dynamics:**

$\Rightarrow$ **There is no guarantee the new computed state is physically accessible from the original 2-D equilibrium via real dynamics**

Nonlinear Initial Value Approach Guarantees Accessibility (with caveats)

- **Full 3-D nonlinear stability calculation converged to a saturated state inherently includes the plasma response in instability case and if external non-axisymmetric field is imposed**
- **Additional advantages:**
  - Full details of dynamics is obtained
- **Disadvantages:**
  - Numerically extremely time consuming
  - Requires all the correct physics included to guarantee convergence to the physically accessible final state:
    - Two-fluids, vacuum, resistive wall, sheared plasma rotation, kinetic effects, resistive and viscous damping, neoclassical effects may all be important
- **Extended MHD codes NIMROD and M3D are suitable:**
  - Most non-ideal effects are included in some form
  - But full simulation to stationary saturated state using complete complement of available physics is not yet feasible

Neither Dynamic or Nearby Equilibrium Approach is Satisfactory in Either Instability or Applied Field Case

- Dynamic evolution approach requires detailed time consuming physics

- Nearby equilibrium approach cannot guarantee convergence to the right final state unless applicable constraints or conserved quantities relating initial and final states are imposed

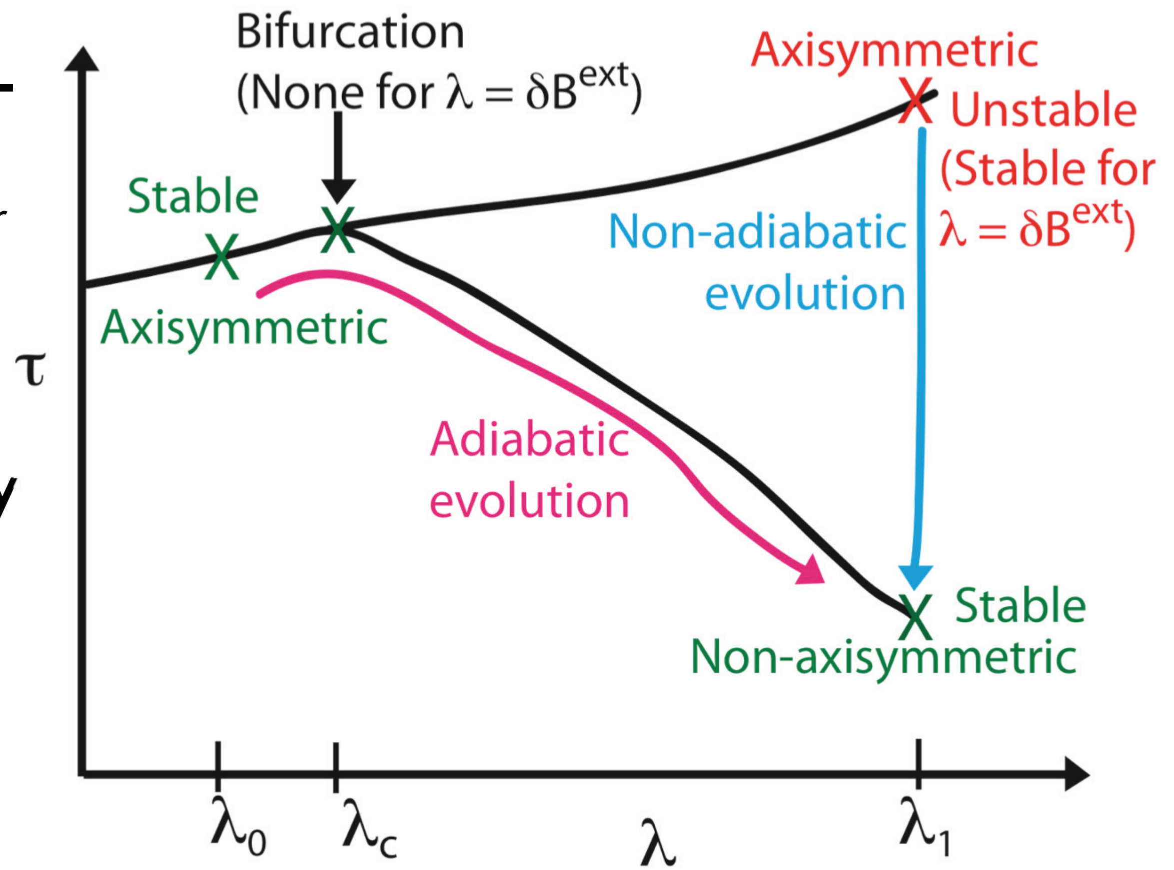
- Can the dynamic approach provide insight into the appropriate constraints required in the nearby equilibrium approach?

  Can a 'Generalized AIMHD' approach be formulated?

Both Generalized AIMHD and Dynamic Instability Evolution Can Also be Applied to Adiabatic Evolution

- **Instead of following the non-adiabatic path one can:**
  - Follow dynamics directly or
  - Construct sequence of neighboring equilibria

  from the initial state as the parameter λ is adiabatically changed
  - In either instability case or external applied case



This may lead to determination of the appropriate set of invariants:

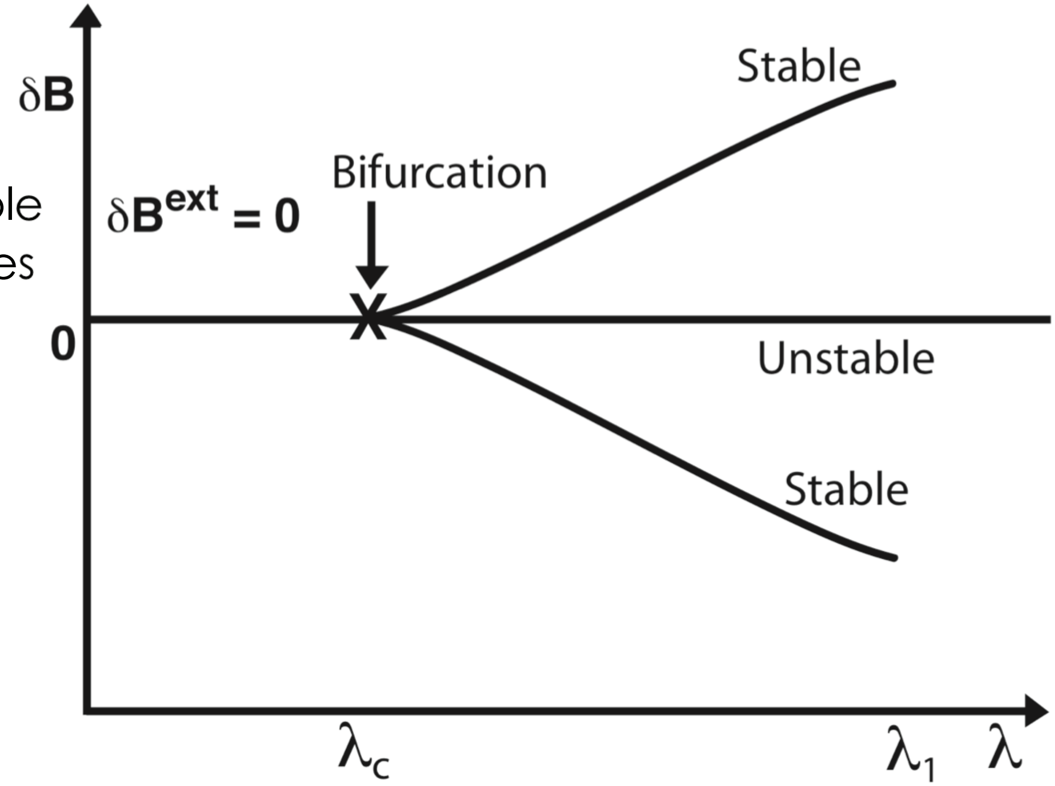
- **A sequence of nearby equilibria needs to be followed but these can be truly nearby:**
  - Linear approach may be applicable

Reiman model: Follow Sequence of Equilibria in Presence of External Non-Axisymmetric Field

- **Reiman picture is valid when dynamics evolves adiabatically through stability boundary in presence of an external non-axisymmetric δB:**
  - i.e. for case with both external field and instability
- **Multiple solutions obtained beyond bifurcation point:**

For $\delta B^{ext} = 0$:

- Axisymmetric state: unstable
- Nonaxisymmetric states: stable
- No preference to stable states



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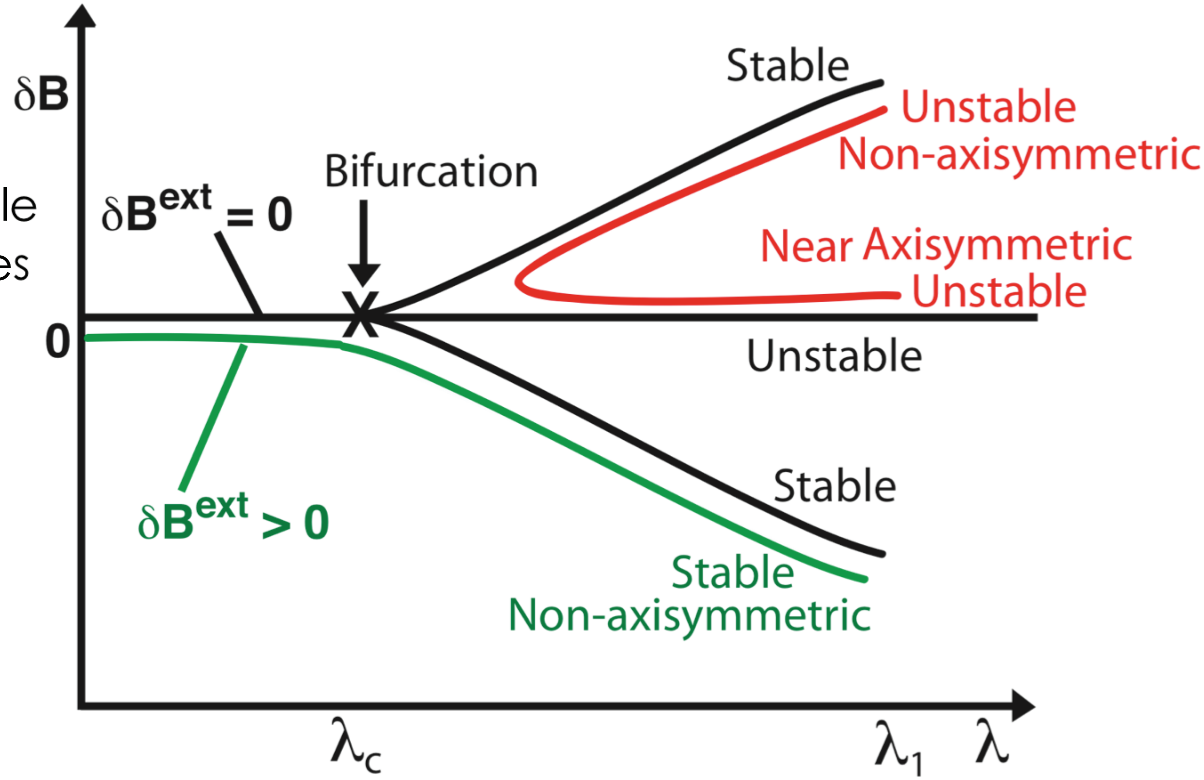


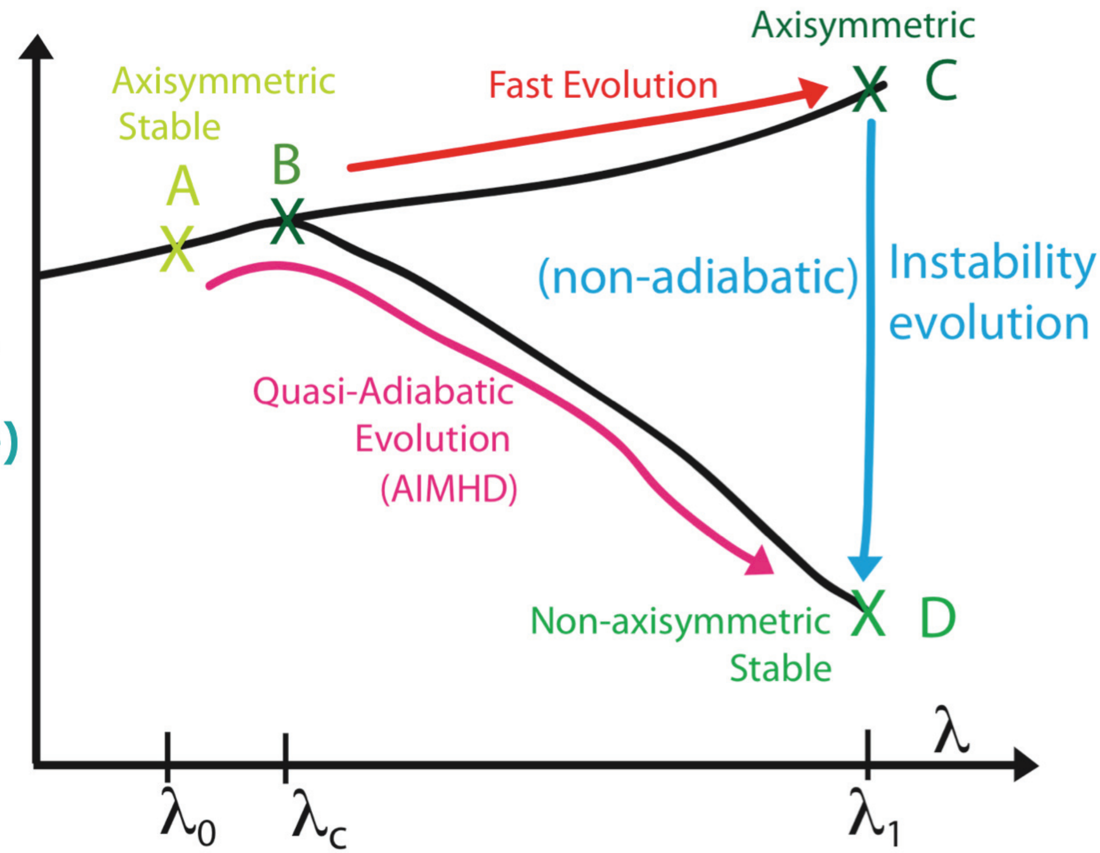
For $\delta B^{ext} > 0$:

- Near axisymmetric: unstable
- Nonaxisymmetric $\delta B^{ext} > 0$: stable
- Nonaxisymmetric $\delta B^{ext} < 0$: stable

$\delta B^{ext} > 0$ removes the bifurcation

Global Picture Shows Relation Between Different Viewpoints

- **Three states are of interest in addition to any bifurcation point:**
  - Initial state A axisymmetric (Stable)
  - Axisymmetric state C (stable or unstable)
  - Final state D nonaxisymmetric (stable)



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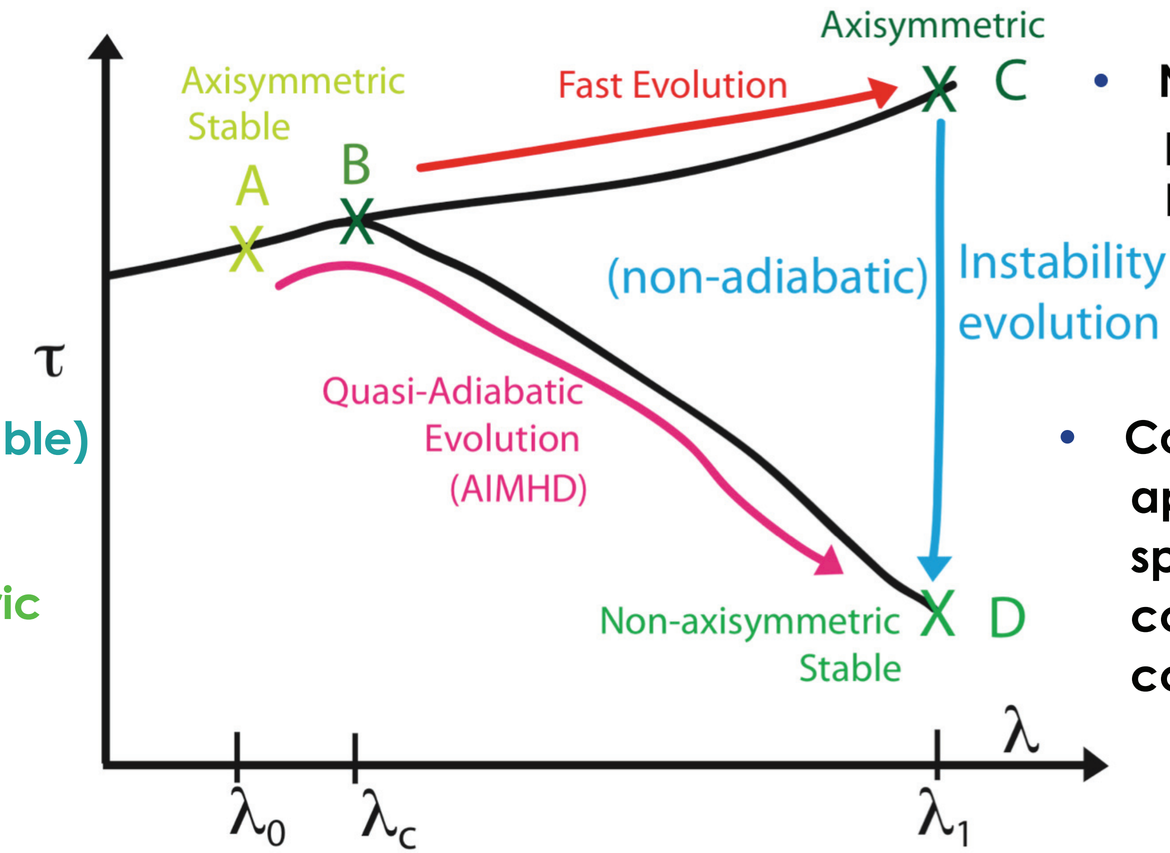

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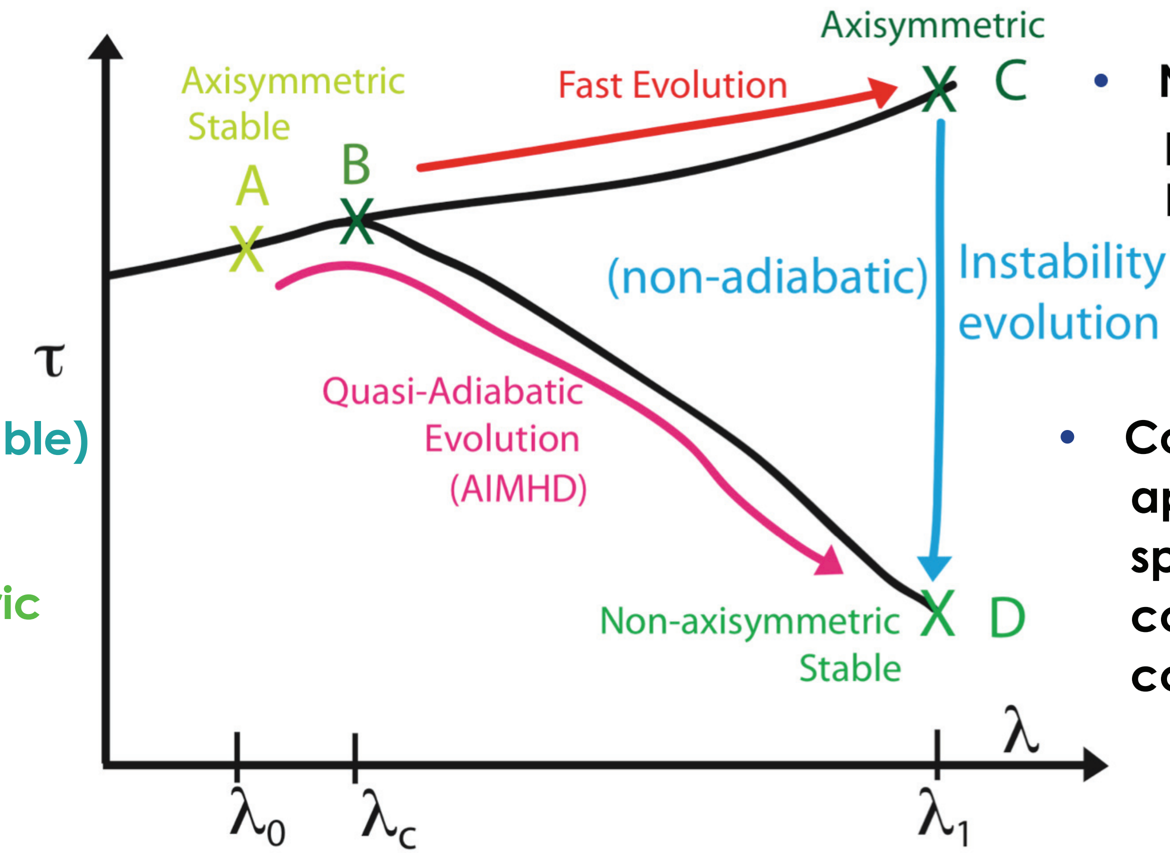
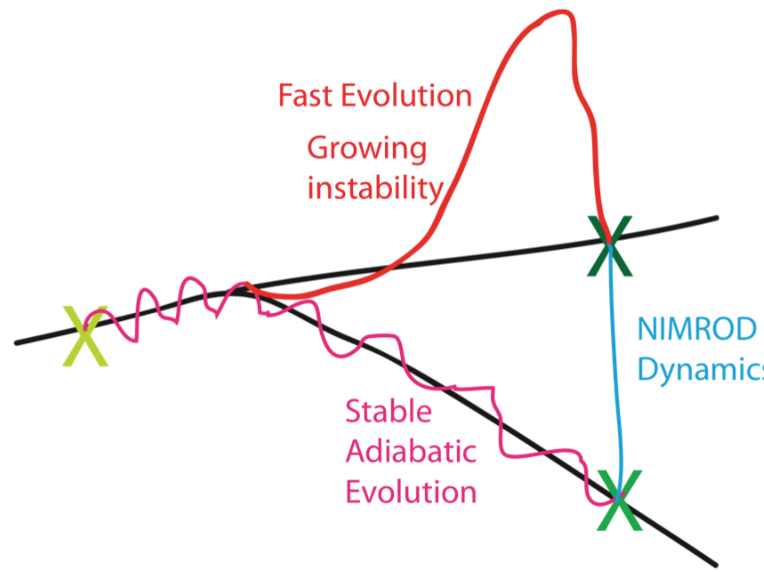


- **Nonlinear stability picture (NIMROD Extended MHD)**
- **Conventional AIMHD approach requires specific assumptions concerning profile constraints**
- **Adiabatic AIMHD approach also requires specific assumptions concerning profile constraints and multiple equilibrium steps**
- **Reiman model is valid when dynamics evolves 'adiabatically' through stability boundary in presence of external applied field**

Constraints Imposed on Equilibrium in AIMHD Need to be Related to Those Imposed During Actual Dynamic Evolution

- **Ultimate goal is to use 'AIMHD' to short circuit full dynamic simulation**
- **AIMHD approach requires specific assumptions for profile constraints:**
  - Generally p = p(ψ,region), where region is a simply connected region isolated from other regions by a separatrix:
    - ⇒ If new regions open up assumptions need to be imposed on p(ψ,region) for those regions
- **AIMHD constraints on flux functions can be informed by dynamic simulations:**
- **Non-adiabatic dynamic path may provide appropriate constraints**
- **Adiabatic dynamic path may provide the key to finding appropriate constraints:**
  - Dynamics through stability boundary can be described as adiabatic Hamiltonian system except near bifurcation point where integrals of motion jump
  - Linear theory may provide insight



3-D Perturbed Equilibria Can be Found From Dynamical Stability or Finding the Nearby Final State Directly

- **Each approach has its advantages:**
  - Dynamical stability problem guarantees convergence to the correct new physical state
  - Perturbed equilibrium problem bypasses the dynamics and converges to a final 3-D state:
- **Each approach has its disadvantages:**
  - Dynamical stability problem requires time consuming numerical calculations with all the physics important for evolution to a new steady state treated correctly:
  - Perturbed equilibrium problem has no guarantee that the correct physical final state is selected:
- **Perturbed equilibrium approach can find the right final state if constraints are imposed:**
  - These constraints need to be informed by a study of the dynamical stability problem