Ideal Ballooning Stability in 3-D equilibrium

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Theses

- The effects of 3-D shaping can impact the stability properties of MHD modes.
  - Many studies in the last decade have been concerned with the effects of 3-D shaping on local eigenvalues (ballooning, microinstabilities)
  - Theoretical tools have been developed to address 3-D shaping --- method of profile variations, local 3-D equilibria
    - Isolate important geometric effects --- curvature, normal torsion, local shear
  - These tools can be applied to understand local stability properties of nearly axisymmetric equilibria perturbed by small 3-D fields (e. g., RMP modification of H-mode equilibria)

Motivation

- There are a number of circumstances where the presence of 3-D fields modify ‘axisymmetric’ toroidal equilibria
  - Field errors, ripple
  - MHD modes
  - Applied control coils (e. g., RMP control of ELMs)
- A number of physical processes at play
  - Topology change (resonant field errors producing magnetic islands)
  - Impact of plasma flow (i. e., NTV damping of toroidal flow)
  - 3-D shaping of magnetic fluxes --- affect stability

The result of shielded applied resonant magnetic perturbations (RMP) is a 3-D distorted equilibrium

- Initial motivation of the applied RMP was to affect edge MHD stability by producing overlapped magnetic islands that flatten profiles--- characterized by Chirikov overlap parameter (Evans et al '07).
  - Rotation shields field error penetration

$$B_r(r_s) \approx \frac{B^V}{(\omega\tau_L)^\alpha} << B^V$$

$$B_\perp \big|_{r_s-}^{r_s+} \approx B^V$$

  - Results is 3-D distorted equilibria with negligibly small islands

3-D equilibria conventionally described by stellarator equilibrium codes

- Understanding properties of 3-D MHD equilibria is crucial for stellarator physics and important in general magnetic confinement
  - Most computational models for 3-D equilibria rely on an MHD model (VMEC, PIES, HINT, SIESTA, etc.) but have different treatments for islands/regions of stochasticity.
  - 3-D MHD stability tools typically assume the existence of flux surfaces.
- Optimizing stability properties with respect to 3-D shaping effects is highly desirous
  - 3-D shaping scans with global codes are largely impractical

3-D shaping scans can be performed for local stability analysis

- 3-D shaping scans can be used for local stability analysis through the use of ‘local 3-D equilibria” (CCH, PoP ‘00).
  - Starting point for conventional ballooning analysis is an evaluation of the ‘local’ eigenvalue --- determined by the properties of a single flux surface ---> Global eigenmode constructed from the properties of local eigenvalues
  - Constructions of sequences of local equilibria
    - Method of profile variations (vary two profile functions)
      - Axisymmetric geometry - Greene-Chance NF ‘81
      - 3-D extension - CCH and Nakajima ‘98
    - Variation of shaping parameters
      - Axisymmetric geometry - Miller et al ‘98
      - Local 3-D equilibria - CCH ‘00

Local 3-D equilibria are specified by the coordinate mapping X(Θ,ζ) and two profiles

- Near the magnetic surface $\psi = \psi_o$, the magnetic coordinates are characterized by the inverse mapping (Θ,ζ) = straight-field line angles

$$\vec{x}(\psi,\Theta,\zeta) = \vec{x}(\psi_o,\Theta,\zeta) + (\psi - \psi_o)\frac{\partial \vec{x}}{\partial \psi}(\psi_o,\Theta,\zeta) + \ldots$$

  - **X** and two profile quantities can be free chosen ---> **X'** is determined by MHD equilibria conditions
  - **J** and **B** are determined by **X**, (ι_o = 1/q at $\psi = \psi_o$)

$$\vec{B} = \frac{\vec{e}_\zeta + \iota_o \vec{e}_\Theta}{\sqrt{g}} = \frac{1}{\sqrt{g}}\left(\frac{\partial \vec{X}}{\partial \zeta} + \iota_o \frac{\partial \vec{X}}{\partial \Theta}\right)$$

$$J^k = \varepsilon_{ijk}\frac{\partial}{\partial \eta_i}\frac{(g_{\zeta j} + \iota_o g_{\Theta j})}{\sqrt{g}} \qquad g_{\zeta\zeta} = \vec{e}_\zeta \cdot \vec{e}_\zeta, g_{\zeta\Theta} = \vec{e}_\zeta \cdot \vec{e}_\Theta, g_{\Theta\Theta} = \vec{e}_\Theta \cdot \vec{e}_\Theta$$

  - Condition **J**·**n** = 0 produces an equation for the Jacobian

$$\frac{\partial}{\partial \Theta}\frac{g_{\zeta\zeta} + \iota_o g_{\zeta\Theta}}{\sqrt{g}} = \frac{\partial}{\partial \zeta}\frac{g_{\Theta\zeta} + \iota_o g_{\Theta\Theta}}{\sqrt{g}}$$

Important geometric quantities are described by X

- Unit normal and tangent vectors are described by **X**(Θ,ζ)

$$\hat{b} = \frac{\vec{B}}{B} = \frac{\vec{e}_\zeta + \iota_o \vec{e}_\Theta}{(g_{\zeta\zeta} + 2\iota_o g_{\zeta\Theta} + \iota_o^2 g_{\Theta\Theta})^{1/2}} \qquad \hat{b}\cdot\nabla = \frac{1}{(g_{\zeta\zeta} + 2\iota_o g_{\zeta\Theta} + \iota_o^2 g_{\Theta\Theta})^{1/2}}\left(\frac{\partial}{\partial\zeta} + \iota_o \frac{\partial}{\partial\Theta}\right)$$

$$\hat{n} = \frac{\nabla\psi}{|\nabla\psi|} = \frac{\vec{e}_\Theta \times \vec{e}_\zeta}{(g_{\zeta\zeta} g_{\Theta\Theta} - g_{\Theta\zeta}{}^2)^{1/2}}$$

- Components of the curvature and torsion vector are calculated from **X**

$$\vec{X} = R(\Theta,\zeta)\hat{R} + Z(\Theta\zeta)\hat{Z} \qquad \phi = \phi(\Theta,\zeta)$$

$$(\hat{b}\cdot\nabla)\hat{b} = \kappa_n \hat{n} + \kappa_g \hat{b}\times\hat{n}$$

$$(\hat{b}\cdot\nabla)(\hat{b}\times\hat{n}) = -\tau_n \hat{n} - \kappa_g \hat{b}$$

$$\kappa_n = \frac{[\phi,z](RR_{\eta\eta} - R^2\phi_\eta^2) + [Z,R](R\phi_{\eta\eta} + 2R_\eta\phi_\eta) + [R,\phi]RZ_{\eta\eta}}{(R_\eta^2 + Z_\eta^2 + R^2\phi_\eta^2)(R^2[\phi,z]^2 + R^2[R,\phi]^2 + [Z,R]^2)^{1/2}}$$

$$\kappa_n = \hat{n}\cdot(\hat{b}\cdot\nabla)\hat{b}$$

$$\kappa_g = (\hat{b}\times\hat{n})\cdot(\hat{b}\cdot\nabla)\hat{b}$$

$$\tau_n = -\hat{n}\cdot(\hat{b}\cdot\nabla)(\hat{b}\times\hat{n})$$

$$R_\eta = \left(\frac{\partial}{\partial\zeta} + \iota_o\frac{\partial}{\partial\Theta}\right)R \qquad [Z,R] = \frac{\partial Z}{\partial\Theta}\frac{\partial R}{\partial\zeta} - \frac{\partial R}{\partial\Theta}\frac{\partial Z}{\partial\zeta}$$

Coordinate mapping X(Θ,ζ) determines |B| and Pfirsch-Schluter current spectrum

- Magnetic field spectrum

$$\frac{1}{B^2} = \frac{(\sqrt{g})^2}{g_{\zeta\zeta} + 2\iota_o g_{\zeta\Theta} + \iota_o^2 g_{\Theta\Theta}}$$

$$|\nabla\psi|^2 = \frac{g_{\zeta\zeta} g_{\Theta\Theta} - g_{\zeta\Theta}{}^2}{(\sqrt{g})^2}$$

- Pfirsch-Schluter coefficient is determined form quasineutrality

$$\vec{J} = \frac{\vec{B}\times\nabla p}{B^2} + p'\lambda\vec{B} + \sigma\vec{B} \qquad \sigma = \frac{<\vec{J}\cdot\vec{B}>}{<B^2>}, \quad <\lambda B^2>=0$$

$$\vec{B}\cdot\nabla\lambda = -\nabla\cdot\frac{\vec{B}\times\nabla\psi}{B^2} \quad \left(\frac{\partial}{\partial\zeta} + \iota_o\frac{\partial}{\partial\Theta}\right)\lambda = 2\frac{|\nabla\psi|}{B}\kappa_g$$

The local magnetic shear can be manipulated by magnetic geometry and plasma profiles

- The local magnetic shear is defined by

$$s = (\hat{b} \times \hat{n}) \cdot \nabla \times (\hat{b} \times \hat{n})$$

- The local shear is related to geometry and profiles by an identity

$$s = \frac{J_{\parallel}}{B} - 2\tau_n = \sigma + p'\lambda - 2\tau_n$$

geometry

profiles

- The local shear can be separated into an average shear and the variation of the local shear

$$s = \frac{|\nabla\psi|^2}{B^2\sqrt{g}}\left[\iota' + \left(\frac{\partial}{\partial\zeta} + \iota_o \frac{\partial}{\partial\Theta}\right)D\right]$$

$$\iota' = \sigma \left< \frac{B^2}{|\nabla\psi|^2} \right> \hat{V}' + p' \left< \frac{\lambda B^2}{|\nabla\psi|^2} \right> \hat{V}' - 2 \left< \frac{\tau_n B^2}{|\nabla\psi|^2} \right> \hat{V}'$$

$$\vec{B}\cdot\nabla D = \sigma\left(\frac{B^2}{|\nabla\psi|^2} - \left< \frac{B^2}{|\nabla\psi|^2} \right> \frac{\hat{V}'}{\sqrt{g}}\right) + p'\left(\frac{\lambda B^2}{|\nabla\psi|^2} - \left< \frac{\lambda B^2}{|\nabla\psi|^2} \right> \frac{\hat{V}'}{\sqrt{g}}\right)$$

$$+ 2\left(\left< \frac{\tau_n B^2}{|\nabla\psi|^2} \right> \frac{\hat{V}'}{\sqrt{g}} - \frac{\tau_n B^2}{|\nabla\psi|^2}\right)$$

Local helical axis equilibria models quasihelically symmetric configuration

- 3-D shaping alter local shear
  - Field line parameterized by

$$R = R_o + \rho_o \cos\theta + \Delta\cos(N\zeta) + \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\sin\theta$$

$$Z = \rho_o \sin\theta + \Delta\sin(N\zeta) - \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\cos\theta$$

  - Important geometric quantities

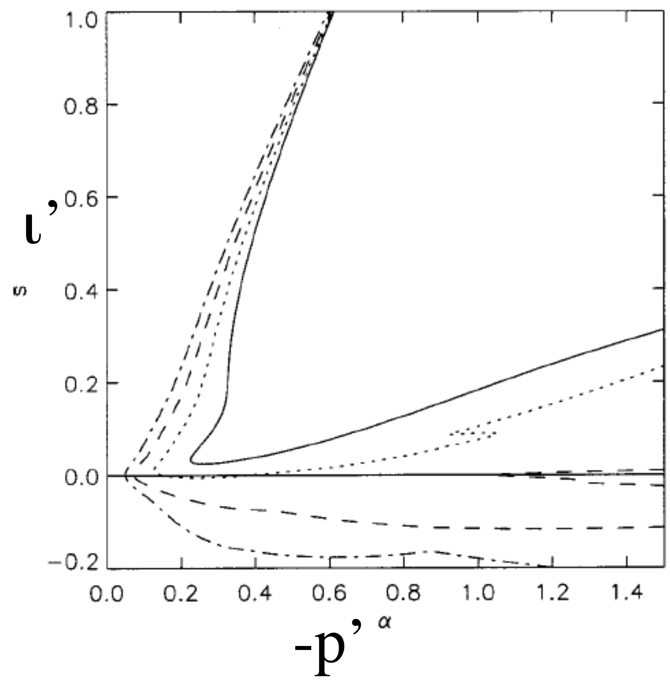
$$\kappa_n \cong -\frac{N^2\Delta}{R_o^2}\cos(N\zeta-\theta)$$

$$\kappa_n \cong -\frac{N^2\Delta}{R_o^2}\sin(N\zeta-\theta)$$

$$\tau_n = \frac{\iota_o}{R_o} - \frac{N^3\Delta^2}{R_o^3}\cos^2(N\zeta-\theta) - \frac{2}{N\Delta}\cos(N\zeta)$$

$$s = \frac{J_\parallel}{B} - 2\tau_n$$

$$\vec{B}\cdot\nabla\frac{J_\parallel}{B} = 2\frac{|\nabla p|}{B}\kappa_g$$



3-D ness

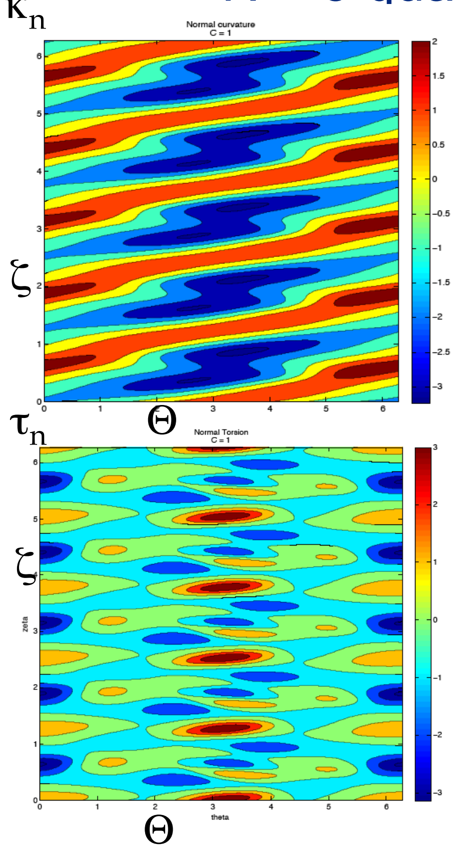
•Instability at s =0 (Anderson localization - Cuthbert and Dewar '00)

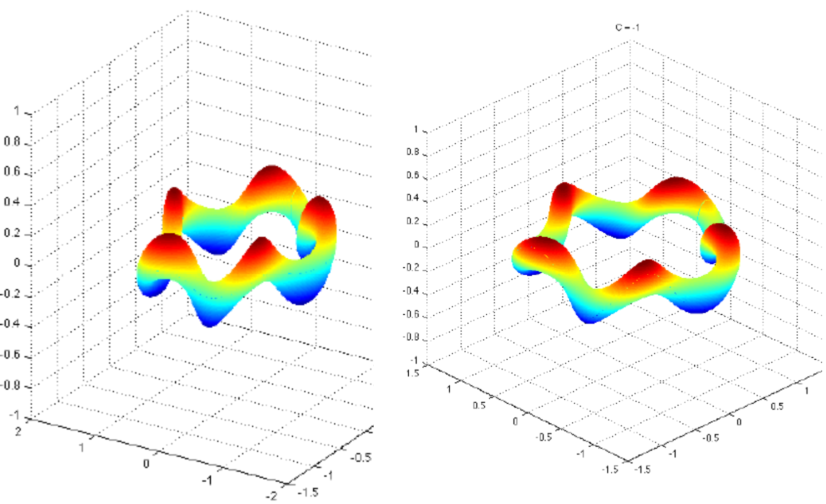
• Loss of second stability (CCH and SRH '03)

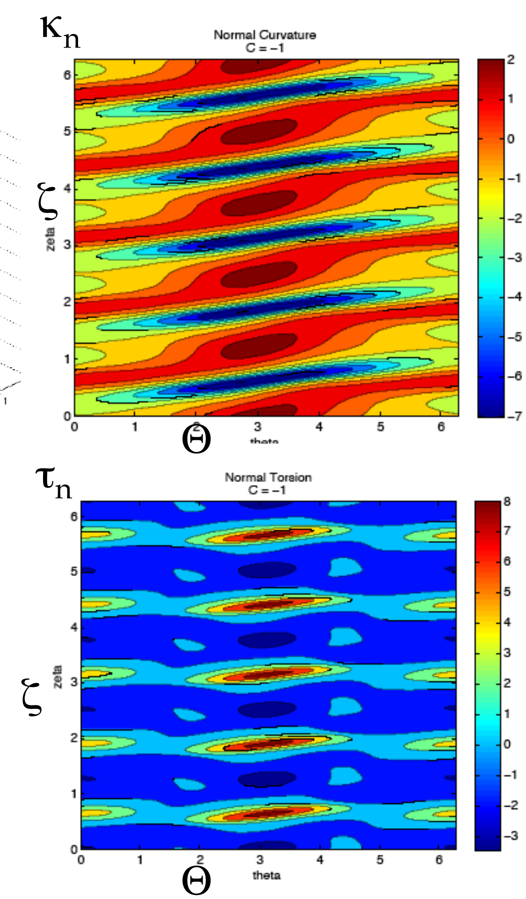
•Field line dependence of eigenvalues

Helical symmetry of helical axis equilibria can be manipulated

- N = 5 quasi-helical equilibria with differing types of symmetry





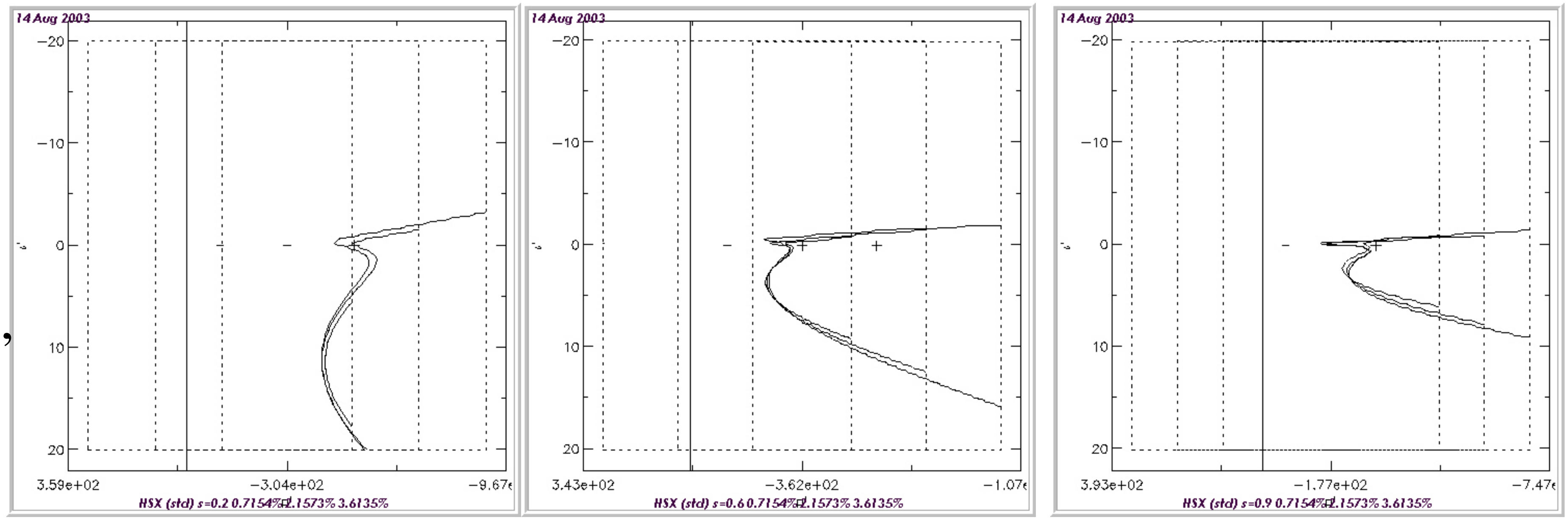


Two different equilibria with varying degrees of toroidal curvature

The quasihelical symmetric configuration does not show second stability

- Stability boundaries for 3 HSX equilibria (β = 0.7%, 2.2%, 3.6 %)

$\psi = 0.2$ $\psi = 0.6$ $\psi = 0.9$



ι'

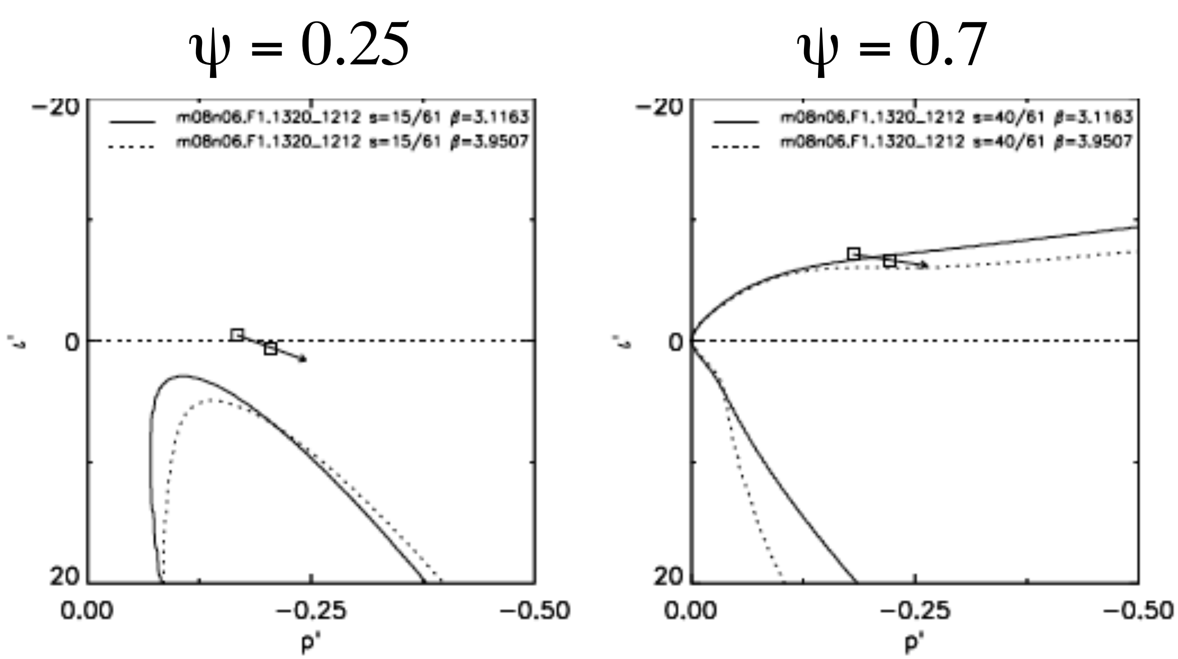
-p' -->

From Hudson et al, PPFC '04

Qualitatively, the same features as the analytic local 3-D equilibria

Analysis of LHD equilibria indicate core second stable

- LHD equilibria characterized by second stable cores and an approach to marginal stability at high β (Nakajima et al NF '07)



Lack of second stable region in the edge

Two high beta LHD equilibria, β = 3.1%, 4.0%

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The effects of a small level 3-D shaping on axisymmetric equilibria can be modeled

- $\mathbf{X}(\Theta,\zeta)$ specified with coordinates $[R,\phi,Z]=[R(\Theta,\zeta),-\zeta,Z(\Theta,\zeta)]$

$$R = R(\Theta) + \sum_{MN} \gamma_{MN} \cos(M\Theta - N\zeta),$$

$$Z = Z(\Theta) + \sum_{MN} \gamma_{MN} \sin(M\Theta - N\zeta)$$

  – 3-D parameter γ controls the level of 3-D shaping relative to an axisymmetric equilibria. In the asymptotic limit ($\gamma \ll \rho_o$)

$$\kappa_n \cong (\kappa_n)_{axi} + \sum_{MN} \frac{M\gamma_{MN}}{\rho_o R_o} \cos(M\Theta - N\zeta)$$

$$\kappa_g \cong (\kappa_g)_{axi} - \sum_{MN} \frac{M\gamma_{MN}}{\rho_o R_o} \sin(M\Theta - N\zeta)$$

$$\tau_n \cong (\tau_n)_{axi} - \sum_{MN} \frac{M[(M-1)\iota_o - N]\gamma_{MN} \cos[(M-1)\Theta - \zeta]}{R_o \rho_o}$$

Small 3-D effect on curvature

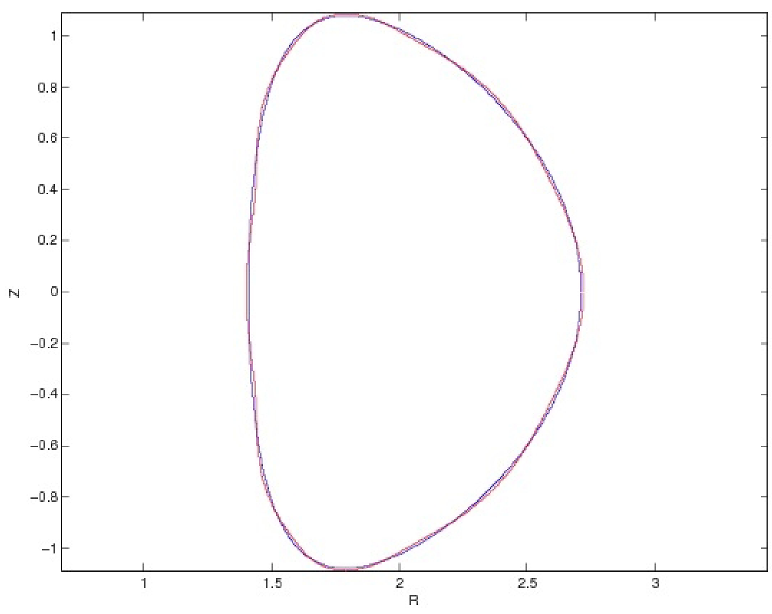
3-D modulation
Of local shear may have
Implications for local
stability

A local 'Miller' equilibria perturbed by small 3-D components can be constructed

- Axisymmetric tokamak (κ = 1.66, δ = 0.416, A= 3.17, q = 3.03) with small γ = 0.01 3-D component (M = 9, N= 3)

$$R = R_o + \rho\cos(\theta - \arcsin\delta\sin(\theta)) + \gamma\sin(M\Theta - N\zeta)$$

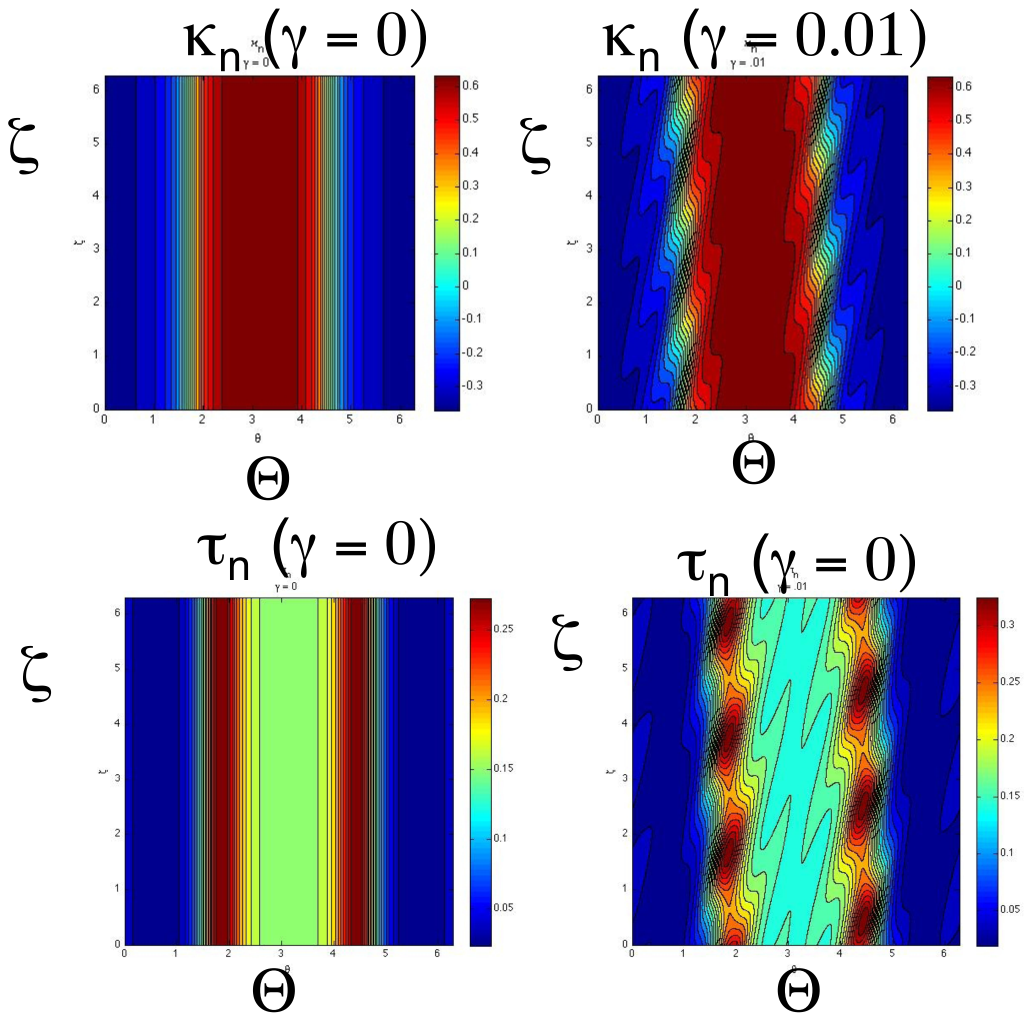
$$Z = \kappa\sin(\theta) + \gamma\sin(M\Theta - N\zeta)$$



Flux surface shapes with γ = 0 and γ = 0.01

3-D perturbation produce small distortion of the flux surface shape

Local shear is sensitive to 3-D fields



- The normal torsion near the outboard midplane is strongly perturbed by 3-D fields --- impacts local shear

- The 3-D fields weakly affect the normal curvature at the outboard midplane

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One can gain analytic insight by calculating the local shear for a shifted circle equilibria

- For shifted circle equilibria (s–α) [s =- rι'/ι, α ~ -p']

$$R = R_0 + \rho[\cos\Theta + \delta\cos(M\Theta - N\zeta)]$$
$$Z = \rho[\sin\Theta + \delta\sin(M\Theta - N\zeta)]$$

  - The integrated local shear

$$\Lambda \sim \int d\Theta\{s - \alpha[\cos\Theta + \delta\frac{M\iota_o}{M\iota_o - N}\cos(M\Theta - N\zeta)] - \delta(\frac{N}{\iota_o} + 1 - M)M\cos[(M-1)\Theta - N\zeta]\}$$

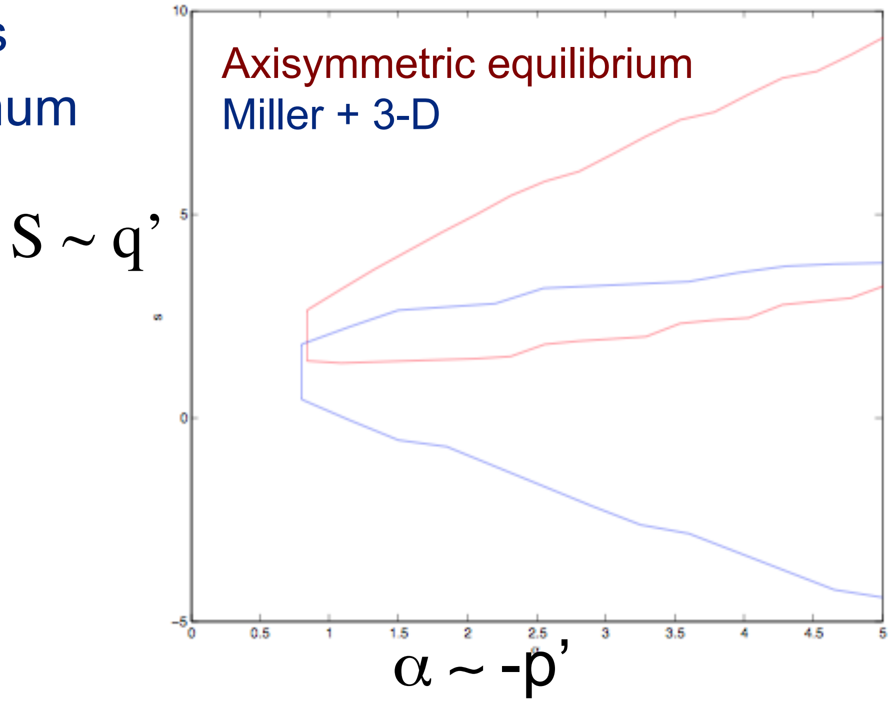
  - Instability tend to reside when local shear is zero
    - 3-D shaping modifies local shear
    - Field line dependent of local Instability eigenvalue

$$\alpha \cong \alpha_c$$

$$\alpha_c = \frac{s - \delta(\frac{N}{\iota_o} + 1 - M)M\cos(N\zeta_o)}{1 + \delta\frac{M\iota_o}{M\iota_o - N}\cos(N\zeta_o)}$$

Preliminary calculations for Miller-like equilibria with 3-D perturbations indicated a deterioration of the second stable region

- Axisymmetric equilibrium (κ = 1.66, δ = 0.416, A= 3.17, q = 3.03)
  - + small γ = 0.01 3-D component (M = 9, N= 3)
  - 3-D calculation uses field line ζ_o = 0 (maximum impact).



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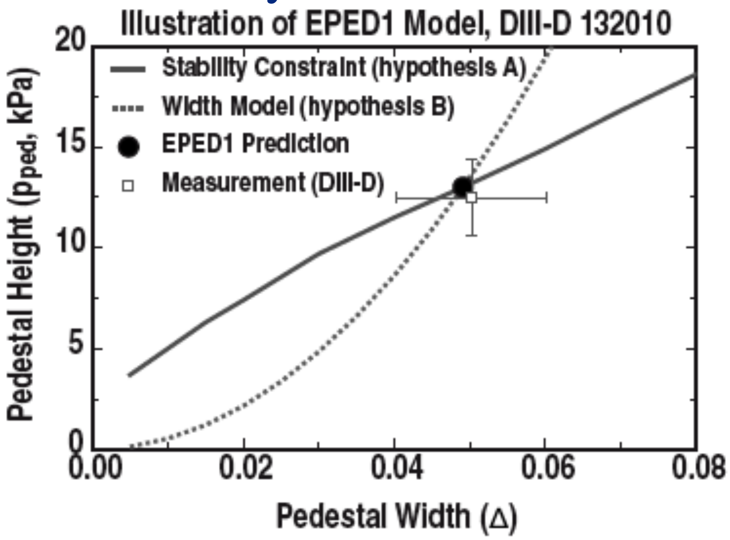
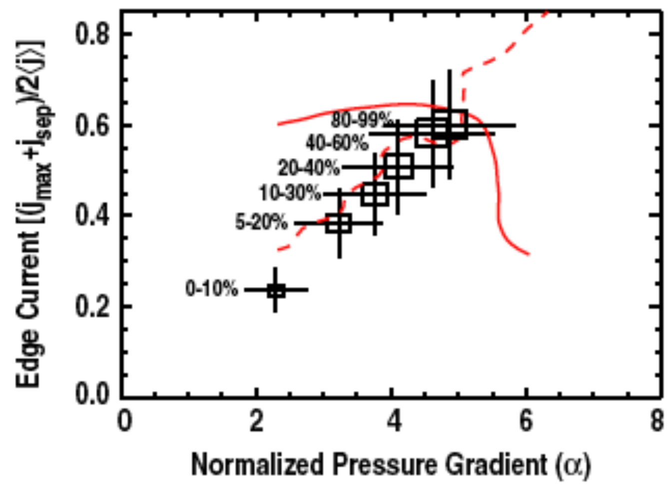
Recent advances in predicting the pedestal width rely on kinetic ballooning and peeling/ballooning model

- EPED1 model has had success in predicting the pedestal width (Groebner et al ‘10). Relies on two elements (Snyder et al ‘09)
  - Intermediate n, peeling-ballooning calculations (ELITE)
  - Assertion that kinetic ballooning modes (KBM) control the transport. KBMs stability largely mirrors ideal MHD stability

From Snyder PoP ‘09



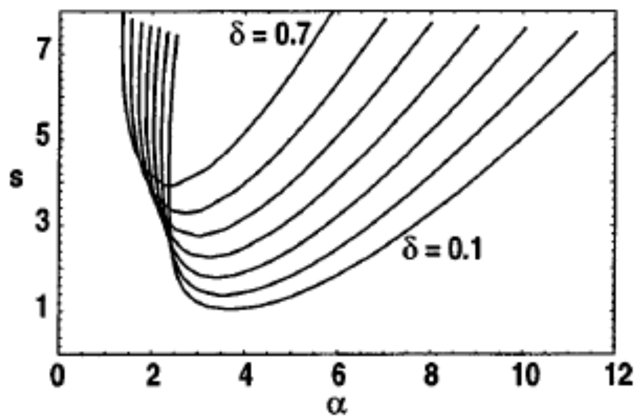
From Groebner NF ‘10



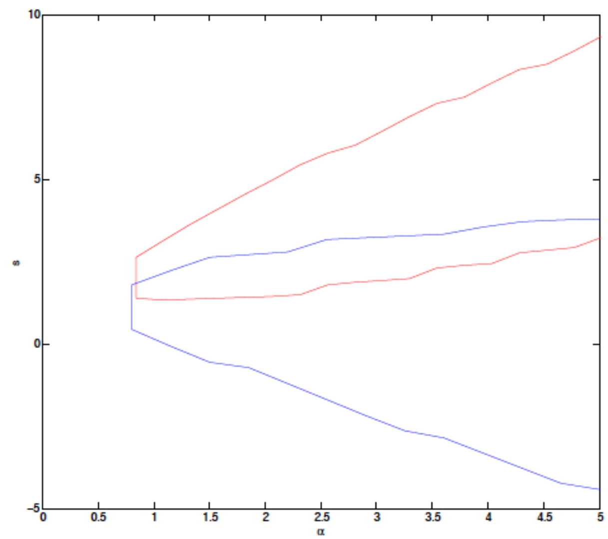
EPED1 scaling is due to a marginal stability scaling argument

- For Miller equilibrium

(Miller et al, PoP '98)



•From 3-D calculation



•Relevant marginal stability points are modeled as $\alpha \sim s^{-1/2}$

---> $\Delta_{ped} \sim (\beta_\theta)^{1/2}$

•3-D perturbations would generally lower this stability boundary

3-D effects can effect both elements of the EPED1 pedestal model

- KBM stability calculated using ideal MHD ballooning code
  - In first stability region, these two calculations are correlated
- Peeling-ballooning modeling (ELITE) relies on an extension of ballooning ordering, describes “global” eigenmode structure
  - Requires Grad-Shafranov solution in the edge region

- 3-D shaping could affect both calculations
  - Correlation of KBM stability to ideal ballooning stability in 3-D equilibrium
  - 3-D equilibrium model with a finite radial extent

Summary

- Theoretical tools have been developed to address the effects of 3-D shaping on local properties --- method of profile variations, local 3-D equilibria --- applied to ballooning stability, microinstability in stellarator
- These tools allow one to Isolate important geometric effects --- curvature, local shear
- These tools can be applied to understand local stability properties of nearly axisymmetric equilibria perturbed by small 3-D fields (e. g., RMP modification of H-mode equilibria)