A New Matching Method for Resistive Wall Mode Analysis of Rotating Plasmas

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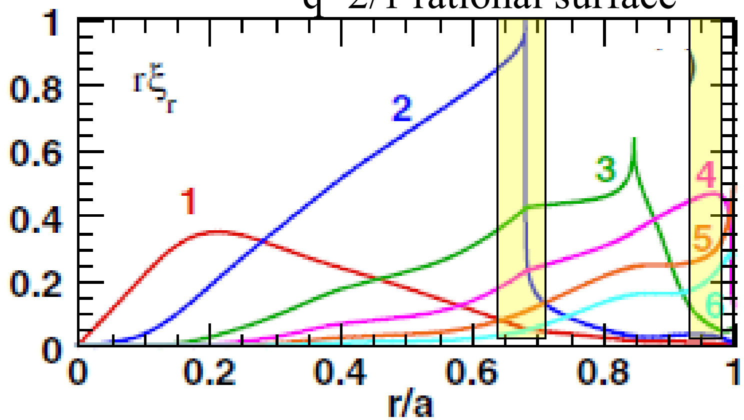
Brief Summary & Motivation

- A new matching method for Resistive Wall Mode (RWM) analysis of rotating plasmas, which resolves some difficulties in existing theories, has been formulated and a preliminary, fast numerical code has been developed.
- The present method
  - retains rotation effects in the vicinity of some surfaces (e.g. rational surface), thus enables the detailed analysis and much save of computation time.
  - can be applied to external modes such as RWMs
  - can be used to study selectively where the rotation effects becomes essential, at rational surface or plasma surface.

q=2/1 rational surface

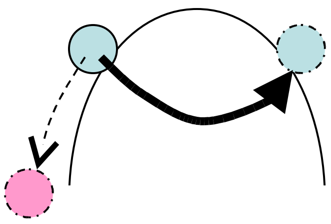


M. Takechi *et al.*,
PRL 98, 055002 (2007)

The plasma displacement of n=1 RWM in JT-60U E46710 discharge.
Numbers denote the poloidal mode number. m=2 is dominant.

Frieman-Rotenberg Equation Governs the Linear Dynamics of Rotating Plasmas

$$\overbrace{\rho\partial_t^2\vec{\xi}}^{\text{inertia}} + \underbrace{\overbrace{2\rho\vec{v}\cdot\vec{\nabla}\partial_t\vec{\xi}}^{\text{Coriolis force}}}_{\text{anti-Hermite}} = \underbrace{\overbrace{\mathcal{F}\vec{\xi}}^{\text{Generalized potential force}}}_{\text{Hermite}}$$



E. Frieman & M. Rotenberg, Rev. Mod. Phys. 32, 898 (1955).

Non-Hermicity → No energy principle exists.[P.J. Morrison (1999)].

We employ an initial value approach.

Some definitions

- $\vec{\xi}$: Lagrangian displacement
- $\vec{v}$: equilibrium rotation
- $\mathcal{F}$: generalized force operator including rotation effects
- $\mathcal{F}\vec{\xi} = 0$: generalized Newcomb equation
- $\mathcal{F}\vec{\xi} = 0$ with $\vec{v} = 0$: static (conventional) Newcomb equation

Hamilton Form of Frieman-Rotenberg Equation

$$\begin{cases} \rho\partial_t\vec{\Pi} = \boxed{\mathcal{F}\vec{\xi} + \rho\vec{v}\cdot\vec{\nabla}\left(\vec{v}\cdot\vec{\nabla}\vec{\xi}\right)} - \rho\vec{v}\cdot\vec{\nabla}\vec{\Pi}, \\ \rho\partial_t\vec{\xi} = -\rho\vec{v}\cdot\vec{\nabla}\vec{\xi} + \rho\vec{\Pi} \end{cases}$$

Momentum vector $\vec{\Pi} = \partial_t\vec{\xi} + \vec{v}\cdot\vec{\nabla}\vec{\xi}$

The new operator $F\vec{\xi} = \mathcal{F}\vec{\xi} + \rho\vec{v}\cdot\vec{\nabla}\left(\vec{v}\cdot\vec{\nabla}\vec{\xi}\right)$

is suitable for numerical computation because the bilinear form related to $F\vec{\xi}$ does not contain $\left|\vec{v}\cdot\vec{\nabla}\vec{\xi}\right|^2$ term that gives a source of big numerical error.

Full Implicit Scheme Enables Bilinear Formalism

To study slowly growing modes such as RWMs, the full implicit scheme has been employed.

$$\begin{cases} \rho\vec{\Pi} + (\Delta t)\rho\vec{v}\cdot\vec{\nabla}\vec{\Pi} - (\Delta t)F\vec{\xi} = \rho\vec{\Pi}_{\rm old} \\ \rho\vec{\xi} + (\Delta t)\rho\vec{v}\cdot\vec{\nabla}\vec{\xi} - (\Delta t)\rho\vec{\Pi} = \rho\vec{\xi}_{\rm old} \end{cases}$$

The above can be solved by bilinear formulation and finite element method

$$\begin{aligned} (\vec{\xi},\vec{\Pi}) + (\vec{\Pi},\vec{\xi}) - (\Delta t)(\vec{\Pi},\vec{\Pi}) \\ +(\Delta t)(\vec{\xi},\vec{v}\cdot\vec{\nabla}\vec{\Pi}) + (\Delta t)(\vec{\Pi},\vec{v}\cdot\vec{\nabla}\vec{\xi}) \\ +(\Delta t)\mathcal{V}(\vec{\xi}) = (\vec{\xi},\vec{\Pi}_{\rm old}) + (\vec{\Pi},\vec{\xi}_{\rm old}) \end{aligned}$$

where

$$(\vec{\eta},\vec{\xi}) := \int_{\rm plasma} \rho\vec{\eta}^{*}\cdot\vec{\xi}d\tau, \quad \mathcal{V}(\vec{\xi}) = -\int_{\rm plasma} \vec{\xi}^{*}\cdot F\vec{\xi}d\tau$$

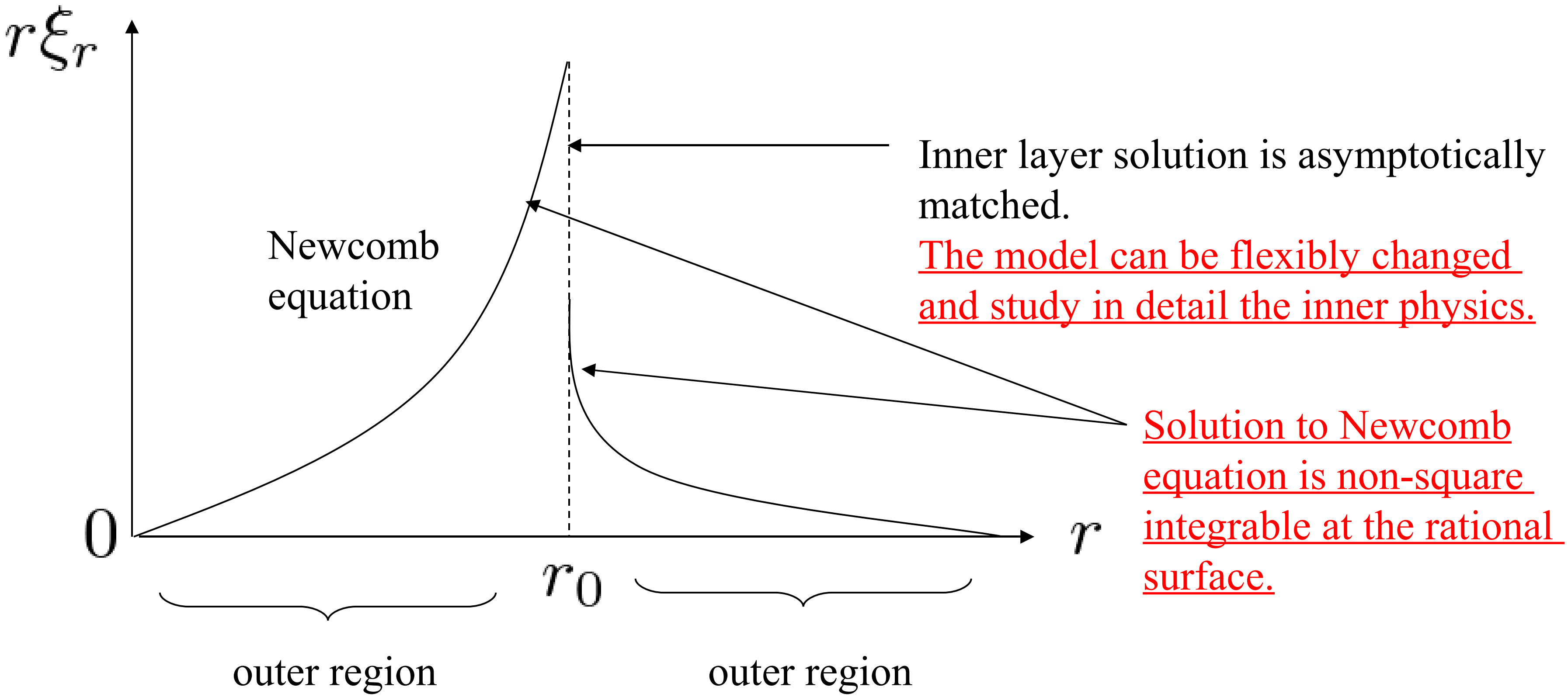
Symmetric Bilinear Form Is Suitable for Finite Element Approximation

$$\mathcal{V}(\vec{\xi}) = \delta W_p(\vec{\xi}) + \delta W_s(\vec{\xi})$$

$$\begin{cases} \delta W_p(\vec{\xi}) & = \displaystyle\int_{\text{plasma}} d\tau \times \\ & \quad \left|\vec{B}\cdot\vec{\nabla}\vec{\xi} - \left(\vec{\nabla}\cdot\vec{\xi}\right)\vec{B}\right|^2 + \Gamma p \left|\vec{\nabla}\cdot\vec{\xi}\right|^2 + \vec{\xi}^* \cdot \left(\vec{\xi}\cdot\vec{\nabla}\vec{\nabla}p_{\text{total}}\right) \\ & \quad + \left[\left(\vec{\nabla}\cdot\vec{\xi}\right)\vec{\xi}^* + \left(\vec{\nabla}\cdot\vec{\xi}^*\right)\vec{\xi}\right]\cdot\vec{\nabla}p_{\text{total}} \\ \delta W_s(\vec{\xi}) & = \displaystyle\int_{\text{surface}} dS \times \left(\hat{n}\cdot\vec{\xi}^*\right)\left(p + \vec{B}\cdot\vec{Q}\right) \end{cases}$$

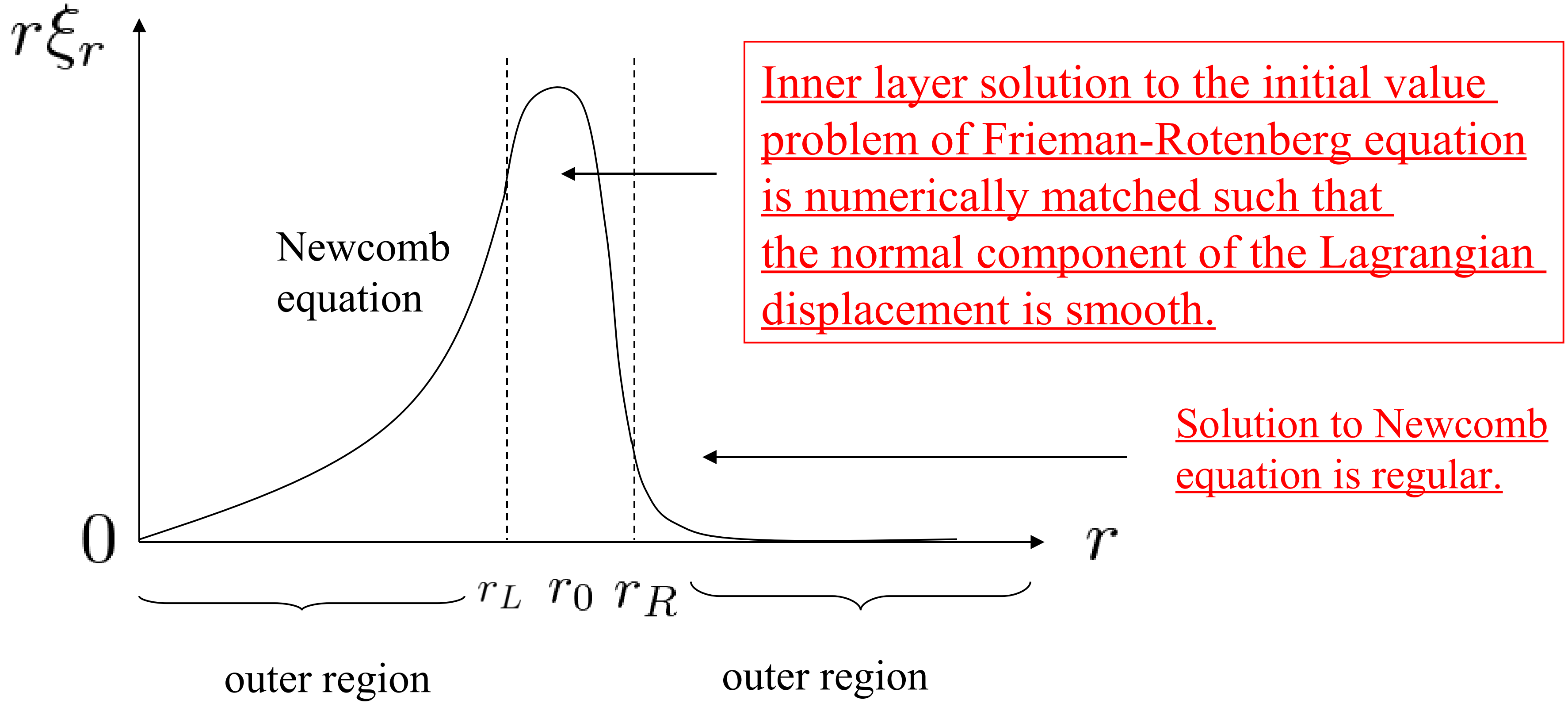
Obviously symmetric form. It's helpful for finite element method (resulting in a large band matrix).

The "Classical (Asymptotic)" Matching Method



- Successful in 1D problem
  - R.D. Hazeltine & J.D. Meiss, *Plasma Confinement*
- Few numerical code for 2D (tokamak) geometry (only PEST?)
  - Needs numerical equilibrium with extremely high accuracy

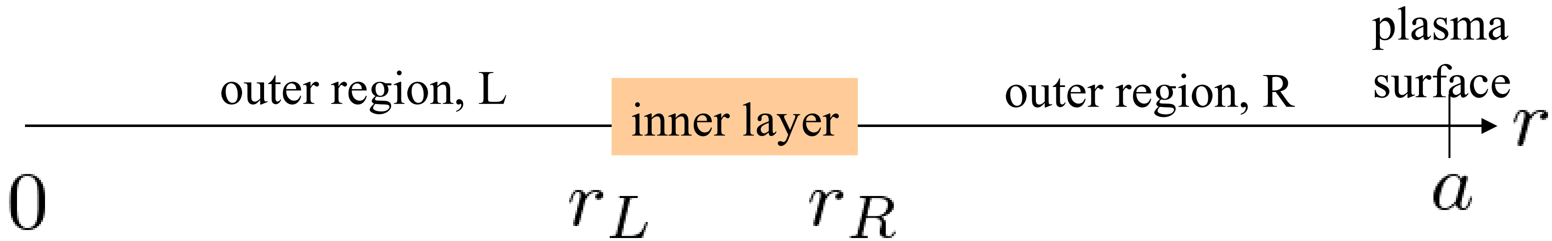
A New Matching Method Resolves the Difficulties



- The inner layer has finite width → Newcomb equation is regular. Thus we have numerically tractable regular solutions in outer regions.
- <u>We have developed a numerical code for cylindrical plasmas to verify the effectiveness of present method.</u>
  - Application to internal kink mode is shown in S. Tokuda, J. Shiraishi, Y. Kagei & N. Aiba, 22nd IAEA FEC TH/P9-20

Detail of New Matching Method (1)

Solutions in outer regions where the inertia can be neglected

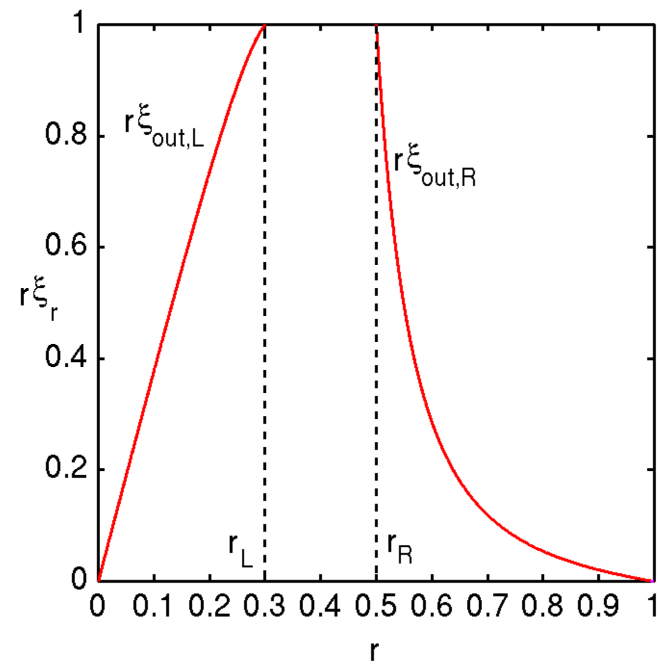


$$\begin{cases} \vec{\xi}_L(r,t) = c_L(t)\vec{\xi}_{\mathrm{out,L}}(r) \\ \vec{\xi}_R(r,t) = c_R(t)\vec{\xi}_{\mathrm{out,R}}(r) \end{cases}$$

Boundary conditions for $\vec{\xi}_{\mathrm{out,p}}$ $(p = L, R)$

$$\begin{cases} r_L \xi_r(r_L) = 1 \quad \text{for} \quad \vec{\xi}_{\mathrm{out,L}} \\ r_R \xi_r(r_R) = 1 \quad \text{for} \quad \vec{\xi}_{\mathrm{out,R}} \end{cases}$$

$c_p(t)$: undermined constants



Detail of New Matching Method (2)

Inner layer solution to full implicit form of Frieman-Rotenberg eq.

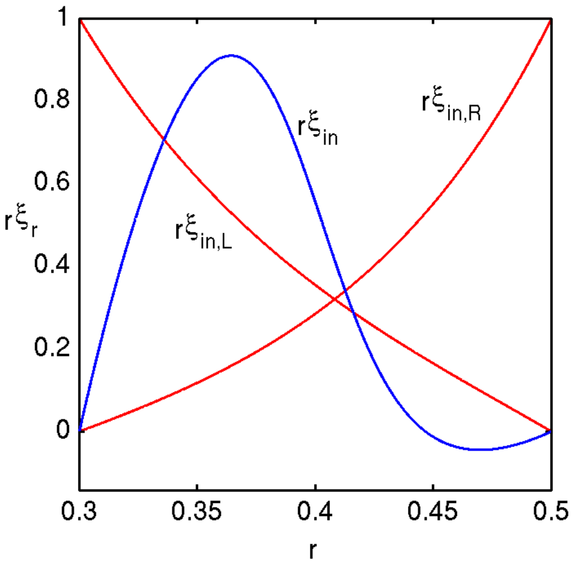
$\vec{\xi}_{\mathrm{in,p}}(r)$ solutions to homogeneous eq. under inhomogeneous boundary condition

$$\begin{cases} r_L\xi_r(r_L) = 1,\ r_R\xi_r(r_R) = 0 & \text{for } \vec{\xi}_{\mathrm{in,L}} \\ r_L\xi_r(r_L) = 0,\ r_R\xi_r(r_R) = 1 & \text{for } \vec{\xi}_{\mathrm{in,R}} \end{cases}$$

$\vec{\xi}_{\mathrm{in}}^{n+1}(r)$ solutions to inhomogeneous eq. under homogeneous boundary condition

General solution is

$$\begin{aligned} \vec{\xi}^{n+1}(r) \quad = \quad & \vec{\xi}_{\mathrm{in}}^{n+1}(r) \\ & + c_L^{n+1}\vec{\xi}_{\mathrm{in,L}}(r) \\ & + c_R^{n+1}\vec{\xi}_{\mathrm{in,R}}(r) \end{aligned}$$



Detail of New Matching Method (3)

Matching condition = radial component of displacement is smooth at matching positions

→ linear simultaneous equation for for each time step c_p^{n+1}

$$A \begin{pmatrix} c_L^{n+1} \\ c_R^{n+1} \end{pmatrix} = \begin{pmatrix} (\xi_{\rm in}^{n+1})'(r_L) \\ (\xi_{\rm in}^{n+1})'(r_R) \end{pmatrix}$$

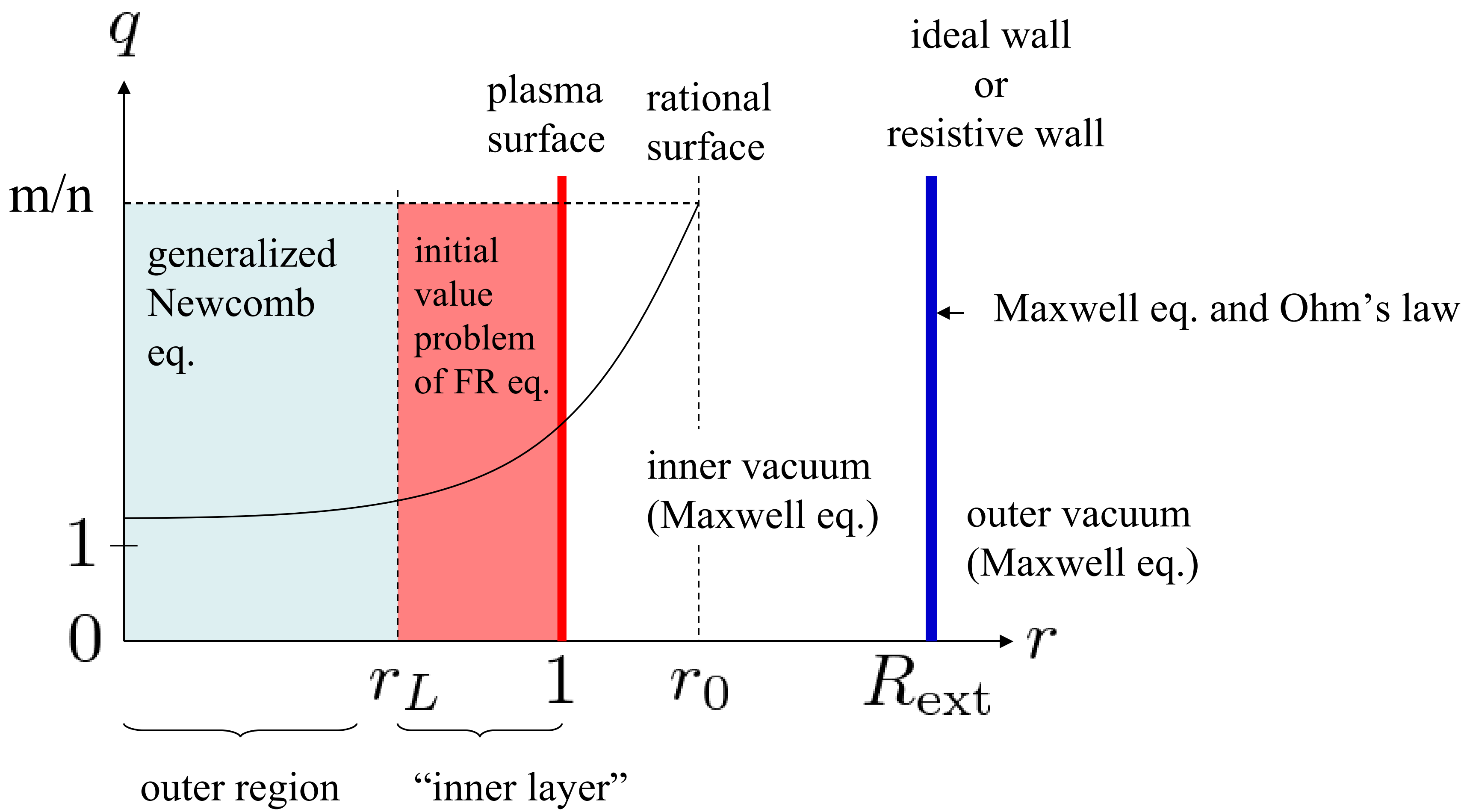
where

$$A = \begin{pmatrix} \xi'_{\rm out,L}(r_L) - \xi'_{\rm in,L}(r_L) & -\xi'_{\rm in,R}(r_L) \\ -\xi'_{\rm in,L}(r_R) & \xi_{\rm out,R}(r_R) - \xi'_{\rm in,R}(r_R) \end{pmatrix}$$

This linear equation is very easy to solve numerically.

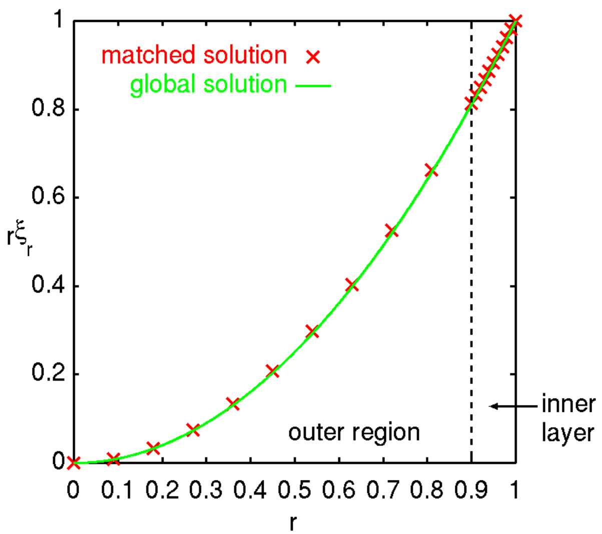
Test : n=1 Ideal External Kink Mode (1)

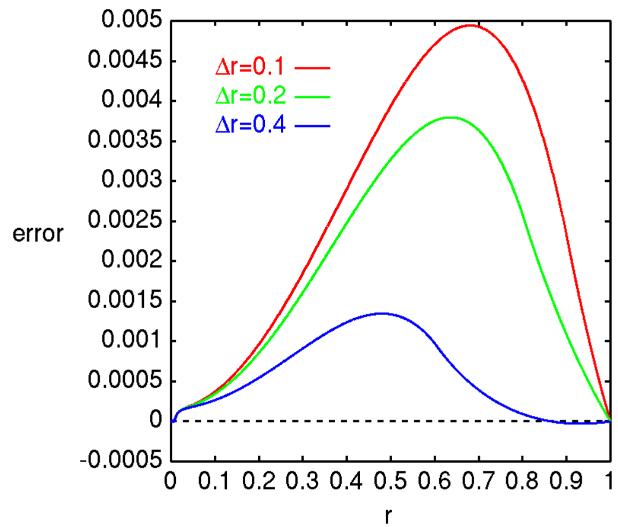
First let us start to study the n=1 ideal external kink.

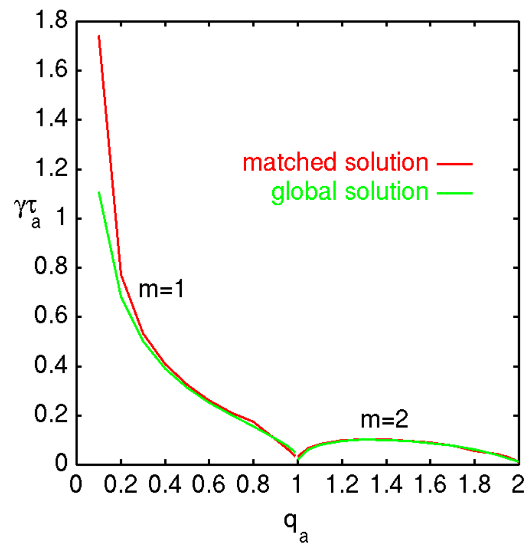


Test : n=1 Ideal External Kink Mode (2)

To test the code, set the wall infinite far and use the “flat current density” equilibrium.







Radial displacement when q_a=1.3. The matched solution agrees well with the global one.

Error between matched solution and global one for some Δr. The error is quite small even when Δr=0.1.

Stability diagram for the uniform current model corresponding to n=1. The inner layer width is Δr=0.1.

Formulation for RWMs (1)

Resistive wall dynamics = pre-Maxwell eq. and Ohm's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}, \quad \partial_t \vec{B} = -\eta \vec{\nabla} \times \vec{j}$$

Current density on resistive wall and vacuum magnetic field

$$\begin{cases} \vec{j}(r,\theta,z,t) = \left[\hat{r} \times \vec{\nabla}\kappa(\theta,z,t)\right] \delta(r-b) & \text{in resistive wall} \\ \vec{B} = \vec{\nabla}\chi^{(-)} & \text{in inner vacuum} \\ \vec{B} = \vec{\nabla}\chi^{(+)} & \text{in outer vacuum} \end{cases}$$

Ampere's law and Faraday's law

$$\begin{cases} \lim_{r \to b} \left[\chi^{(+)}(r,\theta,z,t) - \chi^{(-)}(r,\theta,z,t)\right] = \mu_0 \kappa(\theta,z,t) \\ \partial_t B_r = -\dfrac{\eta}{d} \Delta\kappa \qquad d \text{ : shell width} \end{cases}$$

Formulation for RWMs (2)

With the aid of "thin shell approximation,"
Bilinear form of initial value problem of Frieman-Rotenberg equation and resistive wall dynamics

$$(\vec{\xi},\vec{\Pi})+(\vec{\Pi},\vec{\xi})-(\Delta t)(\vec{\Pi},\vec{\Pi})$$
$$+(\Delta t)(\vec{\xi},\vec{v}\cdot\vec{\nabla}\vec{\Pi})+(\Delta t)(\vec{\Pi},\vec{v}\cdot\vec{\nabla}\vec{\xi})+(\Delta t)\delta W_p$$
$$+(\Delta t)\left[\delta W_{\mathrm{OV}}+\delta W_{\mathrm{IV}}+D_{\mathrm{W}}\right]=(\vec{\xi},\vec{\Pi}_{\mathrm{old}})+(\vec{\Pi},\vec{\xi}_{\mathrm{old}})$$

ADDITIONAL TERMS —(*)

- δW_{IV} : inner vacuum magnetic energy
- δW_{OV} : outer vacuum magnetic energy
- D_{W} : diffusion at resistive wall

Formulation for RWMs (3)

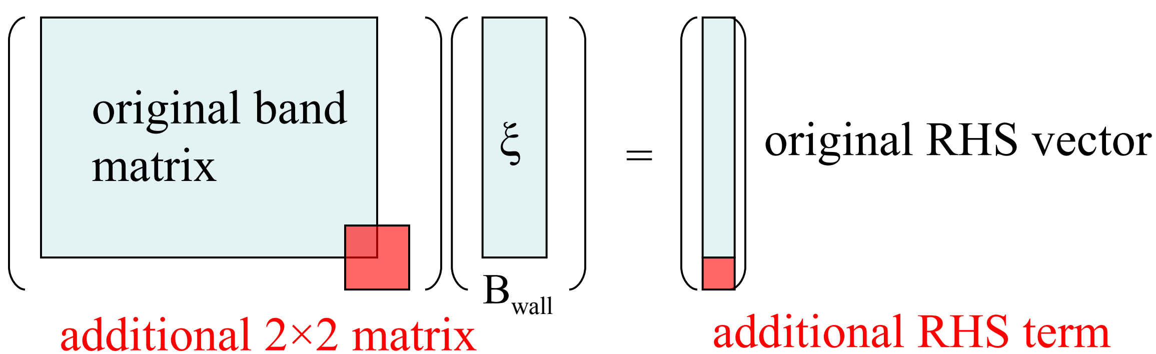
Boundary conditions

The normal magnetic field is continuous at the plasma surface and resistive wall.

$$\begin{cases} \delta W_{\mathrm{IV}} = {}^{T}(\xi_{\mathrm{a}}^{*}, B_{\mathrm{wall}}^{*}) V \begin{pmatrix} \xi_{\mathrm{a}} \\ B_{\mathrm{wall}} \end{pmatrix} \\ \delta W_{\mathrm{OV}} \propto |B_{\mathrm{wall}}|^2 \\ D_W \propto |B_{\mathrm{wall}}|^2 \end{cases}$$

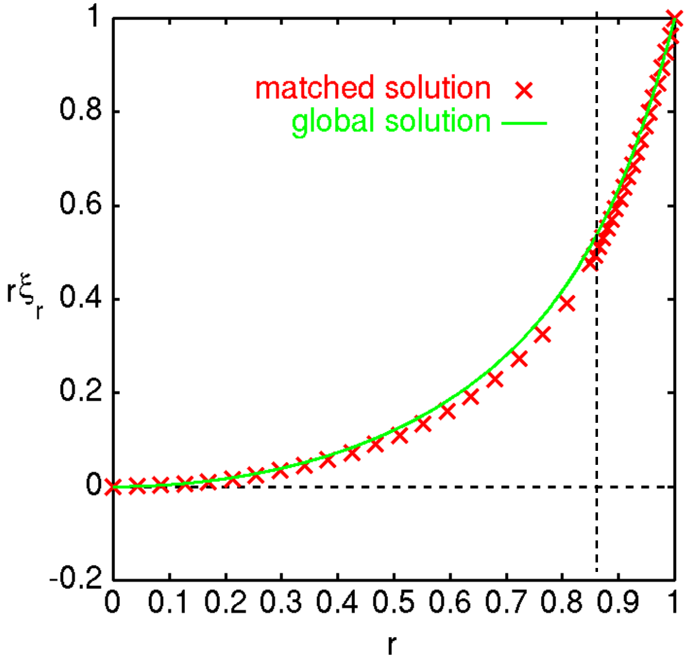
$B_{\mathrm{wall}} = B_r(b)$ is the additional degree of freedom.

New band matrix

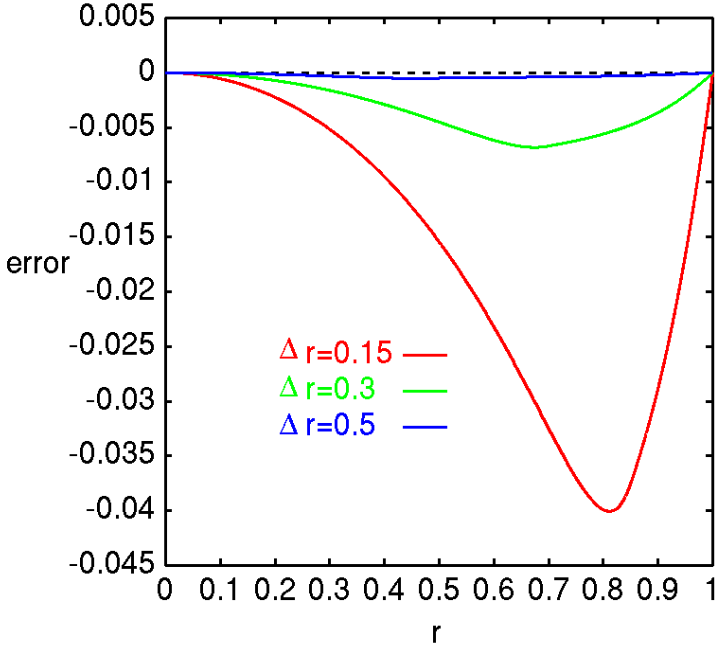


Application : n=1 RWM (1)

Let us assume rigid poloidal rotation (rotation speed at plasma surface=4.6e-4va).
Setting Rext=1.05 can stabilize the external kink mode,
but finite resistivity allows the RWM.

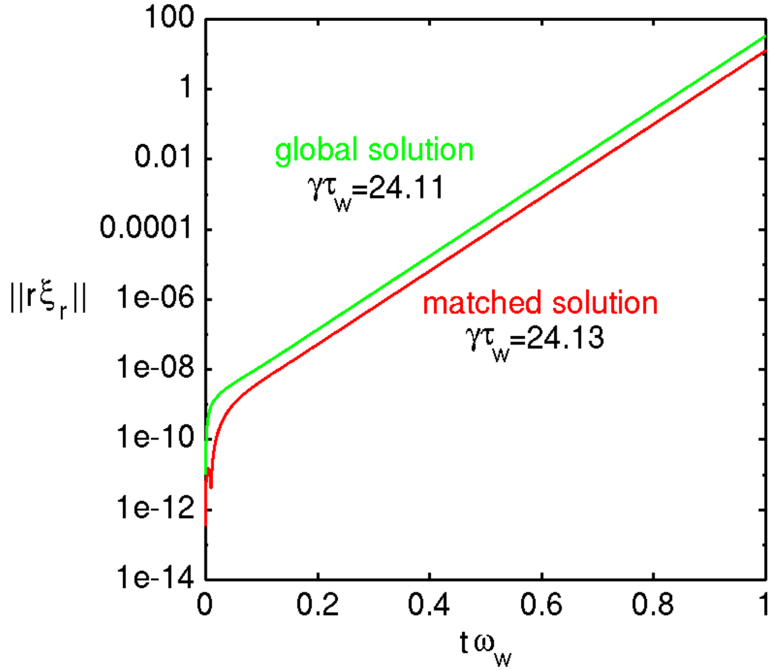


Radial displacement of matched and global solutions when Δr=0.15. Both agree well.

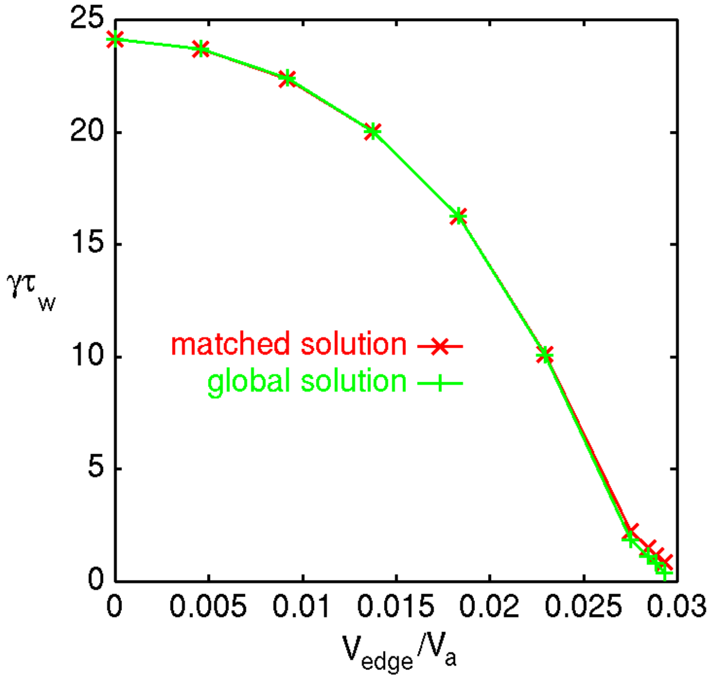


Error between matched solution and global one for some inner layer width. The outer region is modeled by generalized Newcomb equation.

Application : n=1 RWM (2)



Growth rate of matched and global solutions for Δr=0.15. Both agree quite well.



RWM growth rate vs. poloidal rotation speed at plasma surface. As for matching analysis, we have chosen Δr=0.4 to remove the resonant surface in the outer region.
The outer region is modeled by generalized Newcomb equation.

Development of 2D Code without Rotation (RWMaC) (1)

- RWMaC
  - is based on MARG2D, ideal MHD stability code
    - S. Tokuda and T. Watanabe, PoP 6, 3012 (1999).
  - is for tokamak geometry
  - has not yet contained rotation effect

Assuming that the plasma inertia can be neglected in the whole plasma region, the bilinear form can be obtained by setting $\Delta t \longrightarrow \infty$ in (*)

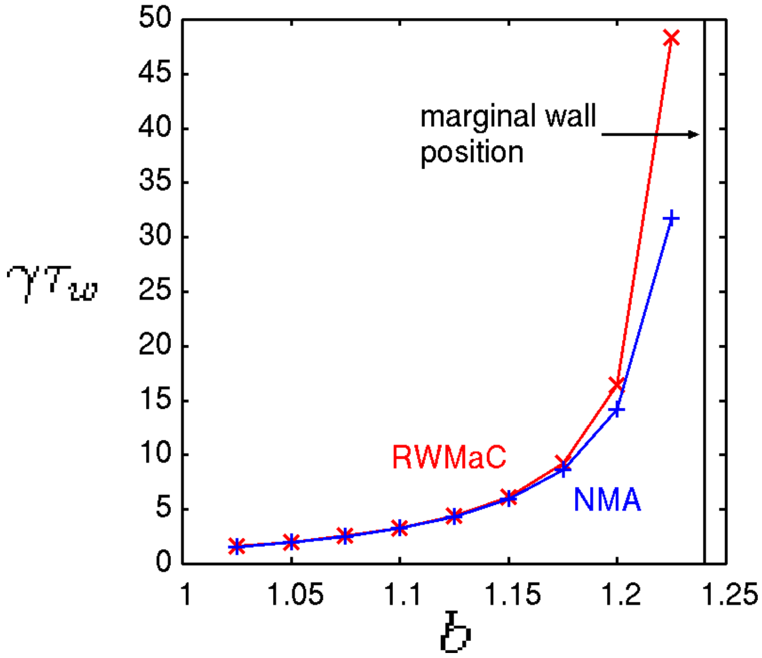
$$\delta W_{\mathrm{P}} + \delta W_{\mathrm{IV}} + \delta W_{\mathrm{OV}} + D_{\mathrm{W}} = 0$$

D. Pfirsch and H. Tasso, NF 11, 259 (1971).

Development of 2D Code without Rrotation (RWMaC) (2)

•RWMaC is benchmarked against NMA (Normal Mode Approach) code [M.S. Chu *et al*., NF 43, 441 (2003).] by studying the Solov'ev equilibrium

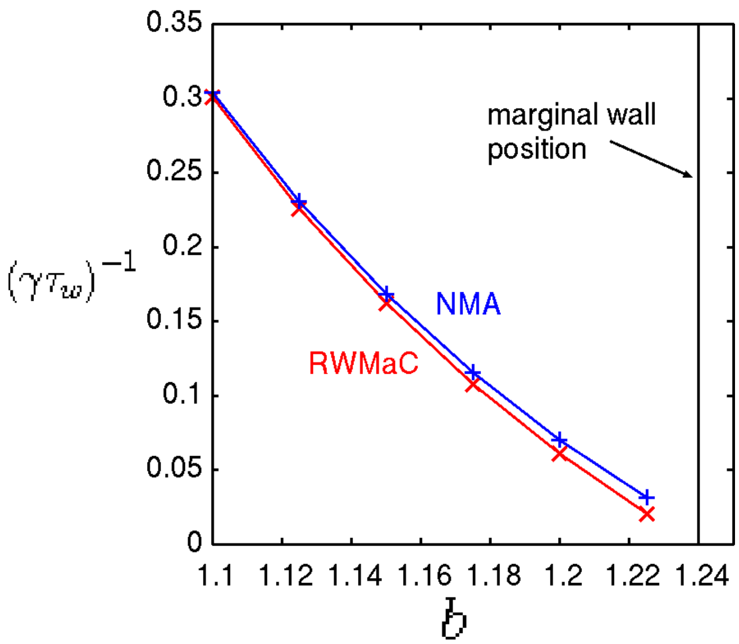
RWM growth rate vs. wall position b



RWM growth rates agree quite well
when the wall is not near the marginal position.

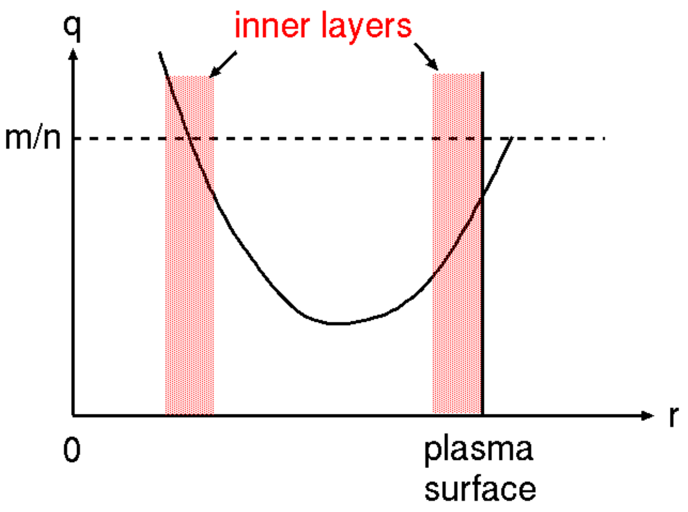
How the RWM growth rate diverges when approaching b→b_{crit} ?

– Inverse of RWM growth rate vs. b



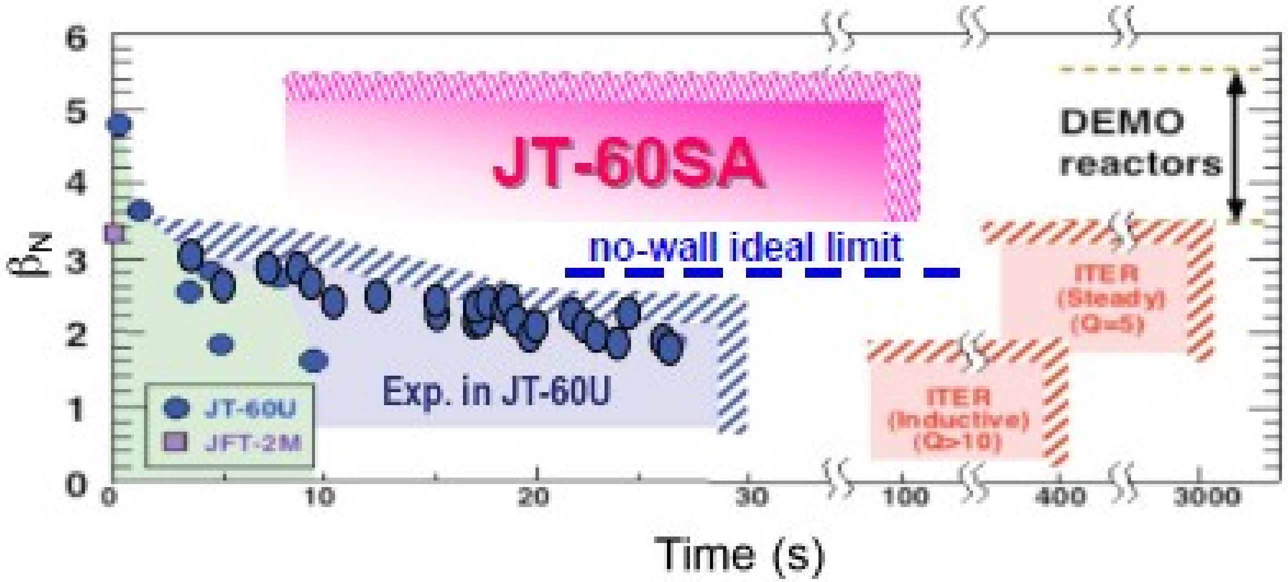
Summary

- We have …
  - proposed a new matching method for RWM analysis of rotating plasmas
  - developed a time-saving numerical code and verified the effectiveness of present method
- We will …
  - study the rotation and rotation shear effects on RWMs in detail
  - study the RWMs in reversed shear plasmas to clarify in which layer the rotation effects are significant, rational surface or plasma surface
  - generalize the method to tokamak geometry (for low-n mode such as RWM, the inner layer located at plasma surface and one more at focused rational surface are sufficient)
  - include kinetic effects (especially trapped particle effect) by perturbation method



Motivation – Why RWMs?

- One of the most important physics issues in advanced tokamak regime is the stabilization of RWMs.



- Experiments have verified that the plasma rotation is promising for RWM stabilization.
  - E.J. Strait *et al.*, PRL 74, 2483 (1995).
  - M. Takechi *et al.*, PRL <u>98</u>, 055002 (2007)
  - H. Reimerdes *et al.*, PRL <u>98</u>, 055001 (2007).

MHD Theory Predicts the Stabilization by Plasma Rotation

- Stabilization of MHD modes by plasma rotation
  - Internal kink modes
    - F.L.Waelbroeck, Phys. Plasmas 3 1047 (1996).
  - RWMs (rotation,kinetic effects + continuum damping)
    - A. Bondeson & D.J. Ward, PRL 72, 2709 (1994).
    - M.S. Chu *et al*., PoP 2, 2236 (1995).
- Theoretically, the critical rotation speed for RWM stabilization is the controversial issue.
  - Rotation / Rotation shear at rational surface [G. Matsunaga *et al*., IAEA FEC Ex/5-2(2008).] ? or rotation effects at plasma surface
  - Which continuum damping?
  - The present method can be applied to analyze the above queries.

MHD Theory for Rotating Plasmas Has Some Essential Difficulties

- (2D) Equilibrium
  - Alfven singularity & Hyperbolicity
    - E. Hameiri PoF 26 230 (1983).
- Linear stability
  - Non-Hermicity → No energy principle exists.
    - P.J. Morrison, Stud. In Appl. Math. 102, 309 (1999).
- We invoke the initial value problem of ideal MHD model.
  - The linear dynamics is governed by the Frieman-Rotenberg equation.
    - E. Frieman & M. Rotenberg, Rev. Mod. Phys. 32, 898 (1955).

Hybrid Finite Element for Cylindrical Plasmas

Since
$$\vec{\nabla}\cdot\vec{\xi} = r^{-1}\left[\frac{d}{dr}(r\xi_r) + m(i\xi_\theta) + k(ir\xi_z)\right] = r^{-1}\left(\frac{dX}{dr} + mY + kZ\right)$$

following non-polluting finite elements have been employed

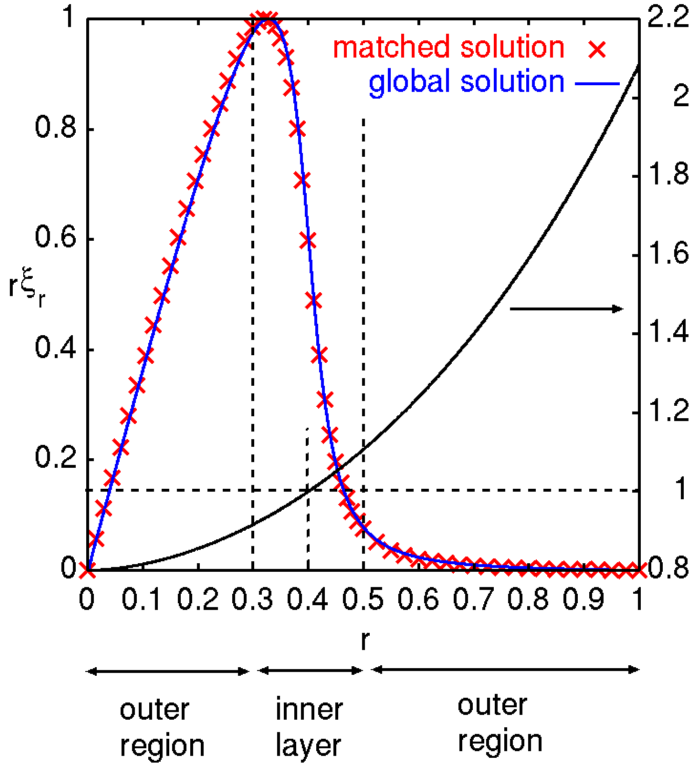
$$\begin{cases} \dfrac{dX}{dr}(r) = \displaystyle\sum_{j=1}^{N_r} \frac{X_{j+1}-X_j}{r_j - r_j} e_{j+1/2}(r), \quad X(r) = \displaystyle\sum_{j=1}^{N_r} \frac{X_{j+1}+X_j}{2} e_{j+1/2}(r) \\ \vec{Y}(r) = \displaystyle\sum_{j=1}^{N_r} \vec{Y}_{j+1/2} e_{j+1/2}(r) \qquad \vec{Y}^T = (Y, Z, U, V, W) \quad \vec{\nabla}\cdot\vec{\Pi} = r^{-1}(dU/dr + mV + kW) \end{cases}$$

where
$$e_{j+1/2}(r) = \begin{cases} 1 & r \in (r_j, r_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

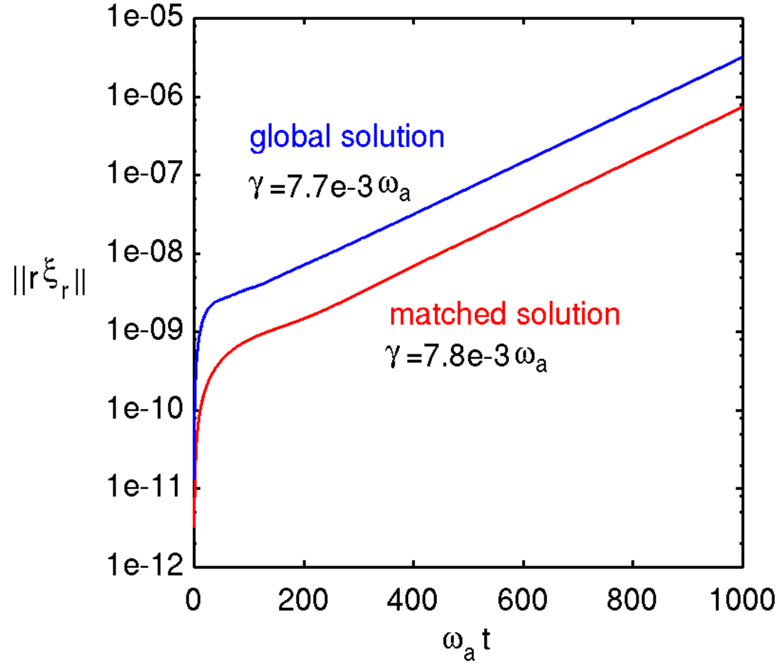
Then, the bilinear form reads $A\vec{x} = \vec{b}$ with $(6N_r + 1)^2$ band matrix

Test : m=1 Internal Kink Mode (1)

Let us assume rigid poloidal rotation and chose as k= -0.2, Δr=0.2, Vedge=0.08v_a

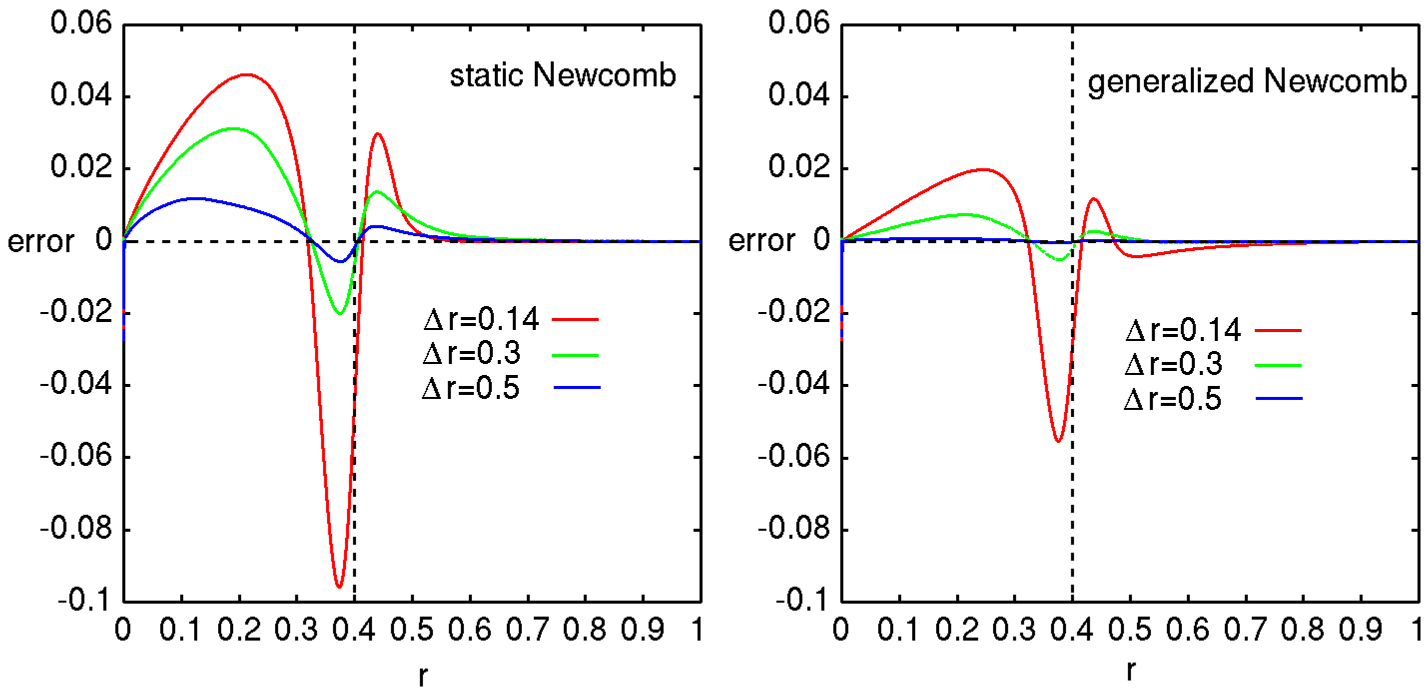


Comparison of radial displacement. The matched solution coincides with the global one.



Comparison of evolution of norm. Growth rate of matched solution agrees well with the global one, but slightly overestimates since matched analysis neglects plasma inertia.

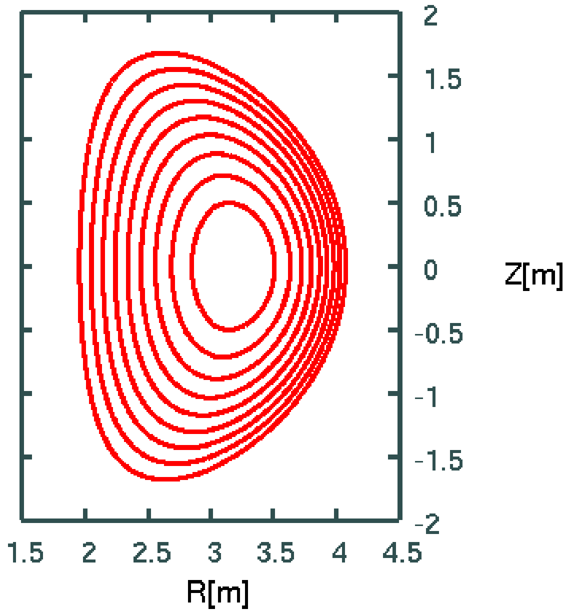
Test : m=1 Internal Kink Mode (2)



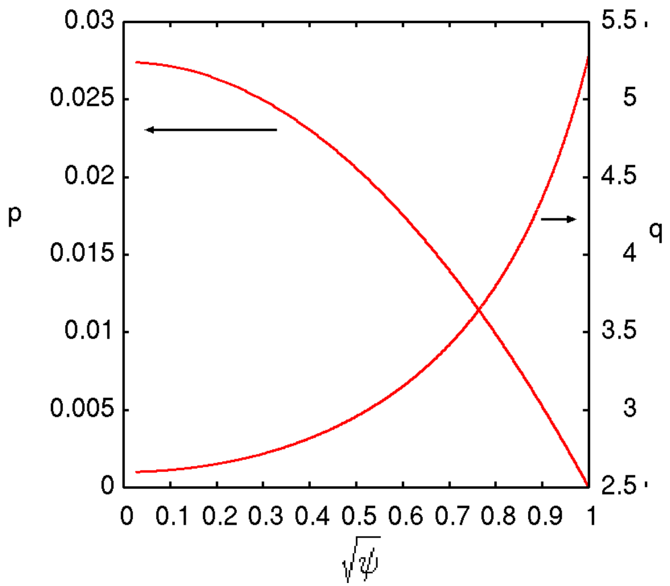
Error between matched solution and global one. The outer regions are modeled by static or generalized Newcomb equation. The error depends on the model and inner layer width. The generalized Newcomb model reduces the error. Errors are quite small even for inner layer width Δr=0.15 for generalized Newcomb model. The error in inside outer region is larger than outside one since magnetic shear is weak there.
The error is asymmetric with respect to the singular surface, which is shifted from the original resonant surface due to plasma rotation.

Development of 2D Code without Rotation (RWMaC) (2)

- RWMaC is benchmarked against NMA (Normal Mode Approach) code by studying the Solov'ev equilibrium



Contour of magnetic poloidal flux

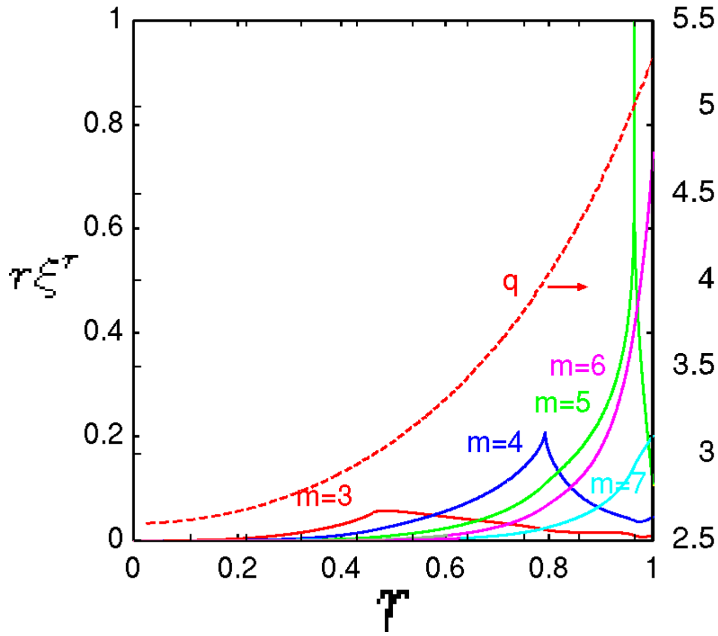


Pressure and safety factor profiles

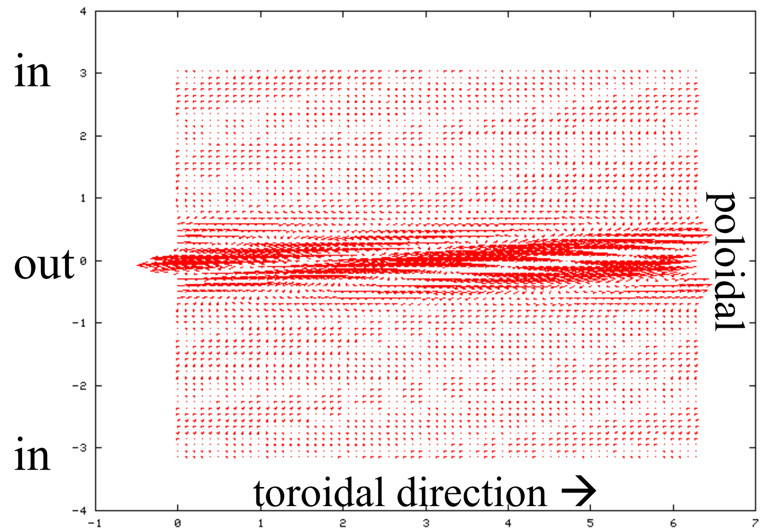
Development of 2D Code without Rotation (RWMaC) (3)

- toroidal mode n=1
- Wall position parameter b := wall radius / plasma radius=1.1

Eigenfunction in the plasma region



Eddy current induced by unstable RWM on the resistive shell



Agrees with NMA