Effects of shear flow on tearing mode stability

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Outline

I. Background and motivation
II. Asymptotic matching
III. Numerical solution to inner-layer equation
IV. Numerical solution in slab geometry without matching as reference
V. Summary

Background and Motivation

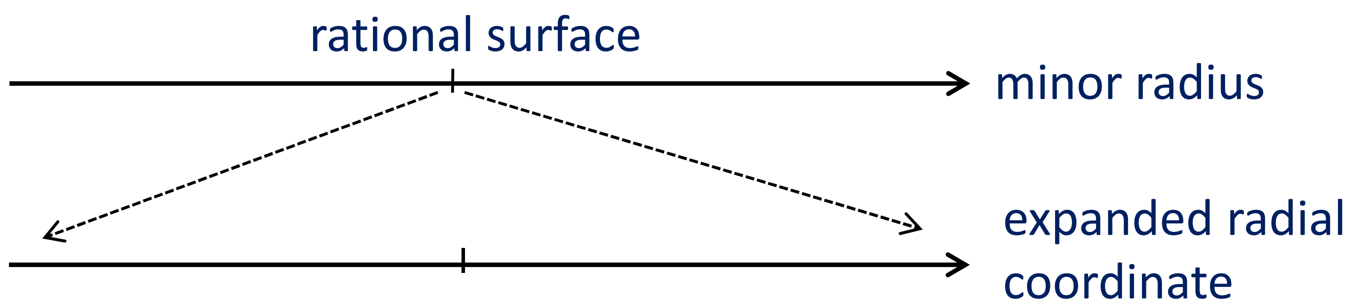
- Tearing mode stability (magnetic islands) in plasmas with sheared rotation, even classical tearing mode as I understand, gets attention experimentally

- Classical tearing mode stability including plasma rotation was studied by, for example,
  - Asymptotic matching theory
    [I. Hofmann, Plasma Phys. **17**, 143 (1975).]
    [R. B. Paris and W. N-C. Sy, Phys. Fluids **26**, 2966 (1983).]
    [X. L. Chen and P. J. Morrison, Phys. Fluids B **2**, 495 (1990).]
  - Global numerical calculation
    [R. Coelho, E. Lazzaro, Phys. Plasmas **14**, 012101 (2007).]
  - Numerical solution to the inner-layer equation
    [S. Tokuda, Nucl. Fusion **41**, 1037 (2001).]

- In this talk,
  - the basis of asymptotic matching technique for rotating plasmas is presented
  - the inner-layer equation is solved numerically by imposing the asymptotic form derived for rotating plasmas

Stability analysis via asymptotic matching technique

- In the stability analysis via asymptotic matching technique, we divide the plasma into
  - Outer region: linearized MHD equation is solved dropping plasma inertia and resistivity (Newcomb equation)
  - Inner region: thin layer around a rational surface where plasma inertia and resistivity are retained and play a role since line-bending becomes weak there



(typical scale length for the expansion is provided by plasma resistivity)

- Following the above consideration, effects of plasma flow might be taken into account through changes of the solution to
  - the Newcomb equation, namely , Δ'
  - the inner-layer equation

Low-beta reduced MHD equations

- We adopt large-aspect-ratio and low-beta reduced MHD equations:

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = -\nabla_{\|} J \\ \dfrac{\partial \psi}{\partial t} + \nabla_{\|}\varphi = \eta J \end{cases}$$

$$\begin{pmatrix} \mathbf{v} = \hat{\mathbf{z}} \times \nabla\varphi \\ \mathbf{B} = \nabla\psi \times \hat{\mathbf{z}} + B_0 \hat{\mathbf{z}} \end{pmatrix}$$

where

$$U = \nabla_{\perp}^2 \varphi \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + [\varphi, \ \ ]$$

$$J = \nabla_{\perp}^2 \psi \qquad \nabla_{\|} := \varepsilon \frac{\partial}{\partial \zeta} - [\psi, \ \ ]$$

$$[f, g] = \hat{\mathbf{z}} \cdot \nabla f \times \nabla g = \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial r} \right)$$

  - Normalization
    - length a, magnetic field B_0, mass density ρ_0 : const.
    - Alfvén velocity $v_{\mathrm{A}} := B_0/\sqrt{\mu_0 \rho_0}$, time $\tau_{\mathrm{A}} := a/v_{\mathrm{A}}$

Outer-region equation

- Linearized low-beta reduced MHD is adopted:

$$\gamma \begin{pmatrix} \nabla_\perp^2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{i}\,m}{r}(v_\theta \nabla_\perp^2 - U_0') & -\mathrm{i}\,m\varepsilon \left(\frac{n}{m} + \frac{1}{q}\right) \nabla_\perp^2 - \frac{\mathrm{i}\,mJ'}{r} \\ -\mathrm{i}\,m\varepsilon \left(\frac{n}{m} + \frac{1}{q}\right) & -\frac{\mathrm{i}\,mv_\theta}{r} + \eta \nabla_\perp^2 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

  where time dependence $\mathrm{e}^{\gamma t}$ is assumed

- In the outer region, the growth rate and resistivity are dropped, while the convection term is retained

$$\begin{cases} -\frac{\mathrm{i}\,m}{r}(v_\theta \nabla_\perp^2 - U_0')\varphi + \left[-\mathrm{i}\,m\varepsilon \left(\frac{n}{m} + \frac{1}{q}\right) \nabla_\perp^2 - \frac{\mathrm{i}\,mJ'}{r}\right]\psi = 0 \\ -\mathrm{i}\,m\varepsilon \left(\frac{n}{m} + \frac{1}{q}\right)\varphi - \frac{\mathrm{i}\,mv_\theta}{r}\psi = 0 \end{cases}$$

- The convection term can change Δ' or the energy available to drive the tearing instability

- In the following, let us consider only the plasma flow which becomes zero at the mode-resonant surface: This can be assumed without loss of generality for fixed boundary conditions

Asymptotic behavior from the outer region

- Substituting

$$\begin{cases} \varphi = \varphi_0 x^\beta \left(1 + \varphi_1 x + \varphi_2 x^2 + \cdots\right) \\ \psi = x^\alpha \left(1 + \psi_1 x + \psi_2 x^2 + \cdots\right) \end{cases}$$

into the outer-region equations, we find that the terms at each order of x nicely balance when $\alpha = \beta$

- The leading order gives us

$$\alpha = 0, 1$$

$$\varphi_0 = -\frac{v'_{\theta \mathrm{s}}}{\varepsilon r_{\mathrm{s}} (1/q)'_{\mathrm{s}}}$$

- At the next order , we find that $\alpha = 0$ cannot be adopted, and we obtain one of the independent solution as

$$\begin{cases} \varphi = \varphi_0 \left(1 + \varphi_1 x + \varphi_2 x^2 + \cdots\right) \\ \psi = 1 + \psi_1 x + \psi_2 x^2 + \cdots \end{cases}$$

- The coefficients of the higher-order terms can be determined successively

Second independent solution

- The outer-region equations can be summarized in a single equation of the form:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + P(r)\frac{\mathrm{d}\psi}{\mathrm{d}r} + Q(r)\psi = 0$$

  which is the same as the Newcomb equation in the absence of plasma rotation

- By examining the coefficients $P(r)$ and $Q(r)$, we find that the mode-resonant surface remains as a regular singularity

- Therefore, we can construct the second independent solution by a conventional method to obtain the Frobenius series solution

- Then the outer solution is expressed as

$$\begin{cases} \varphi = \varphi_0 \left[1 + \left(\psi_1 + \frac{1}{r_\mathrm{s}}\right) x \ln|x| \cdots + A^{\pm}\left(x + \psi_1 x^2 + \cdots\right)\right] \\ \psi = 1 + \left(\psi_1 + \frac{1}{r_\mathrm{s}}\right) x \ln|x| \cdots + A^{\pm}\left(x + \psi_1 x^2 + \cdots\right) \end{cases}$$

Inner-layer equation

- Let us approximate the equilibrium quantities by linear functions around the mode-resonant surface in the reduced MHD equations:

$$\begin{cases} \left(\gamma + \frac{\mathrm{i}\, m v'_{\theta \mathrm{s}}}{r_\mathrm{s}} x\right) \nabla_\perp^2 \varphi = \frac{\mathrm{i}\, m U'_{0\mathrm{s}}}{r_\mathrm{s}} \varphi - \mathrm{i}\, m \varepsilon \left(\frac{1}{q}\right)'_\mathrm{s} x \nabla_\perp^2 \psi - \frac{\mathrm{i}\, m J'_\mathrm{s}}{r_\mathrm{s}} \psi \\ \left(\gamma + \frac{\mathrm{i}\, m v'_{\theta \mathrm{s}}}{r_\mathrm{s}} x\right) \psi = -\mathrm{i}\, m \varepsilon \left(\frac{1}{q}\right)_\mathrm{s} x \varphi + \eta \nabla_\perp^2 \psi \end{cases}$$

- Assuming a stretched coordinate and a frequency by

$$\begin{cases} X = \eta^\alpha X \\ \gamma = \eta^\beta \Gamma \end{cases}$$

and substituting them into the above equations, we find that the convection term, line-bending term and resistive term balance to each other when we choose

$$\alpha = \beta = \frac{1}{3}$$

- Then the inner-layer equations are obtained as

$$\begin{cases} \left(\Gamma + \frac{\mathrm{i}\, m v'_{\theta \mathrm{s}}}{r_\mathrm{s}} X\right) \frac{\mathrm{d}^2 \varphi}{\mathrm{d}X^2} = -\mathrm{i}\, m \varepsilon \left(\frac{1}{q}\right)'_\mathrm{s} X \frac{\mathrm{d}^2 \psi}{\mathrm{d}X^2} \\ \left(\Gamma + \frac{\mathrm{i}\, m v'_{\theta \mathrm{s}}}{r_\mathrm{s}} X\right) \psi = -\mathrm{i}\, m \varepsilon \left(\frac{1}{q}\right)'_\mathrm{s} X \varphi + \frac{\mathrm{d}^2 \psi}{\mathrm{d}X^2} \end{cases}$$

At large X, the convection and line-bending terms becomes dominant, which are retained in the outer-region equation

Asymptotic behavior from the inner layer

- Substituting

$$\left\{\begin{array}{l}\varphi=\Phi_0 X^{-\alpha}\left(1+\Phi_1 X^{-1}+\Phi_2 X^{-2}+\cdots\right)\\ \psi=X^{-\beta}\left(1+\Psi_1 X^{-1}+\Psi_2 X^{-2}+\cdots\right)\end{array}\right.$$

into the inner-layer equations, we find that the terms at each order of nicely balance when

$$\alpha=\beta$$

- The leading order gives us

$$\beta=0,-1$$

$$\Phi_0=-\frac{v'_{\theta s}}{\varepsilon r_s(1/q)'_s}$$

c.f. ratio from the outer solution:

$$\varphi_0=-\frac{v'_{\theta s}}{\varepsilon r_s(1/q)'_s}$$

- By balancing order by order, we obtain

$$\left\{\begin{array}{l}\varphi=\Phi_0\left(1+\cdots+a^{\pm}X+\cdots\right)\\ \psi=1+\cdots+a^{\pm}X+\cdots\end{array}\right.$$

c.f. Outer solution:

$$\left\{\begin{array}{l}\varphi=\varphi_0\left[1+\left(\psi_1+\frac{1}{r_s}\right)x\ln|x|\cdots+A^{\pm}\left(x+\psi_1 x^2+\cdots\right)\right]\\ \psi=1+\left(\psi_1+\frac{1}{r_s}\right)x\ln|x|\cdots+A^{\pm}\left(x+\psi_1 x^2+\cdots\right)\end{array}\right.$$

- The asymptotic behavior of inner solution agrees with the outer solution, therefore, the asymptotic matching technique can be used

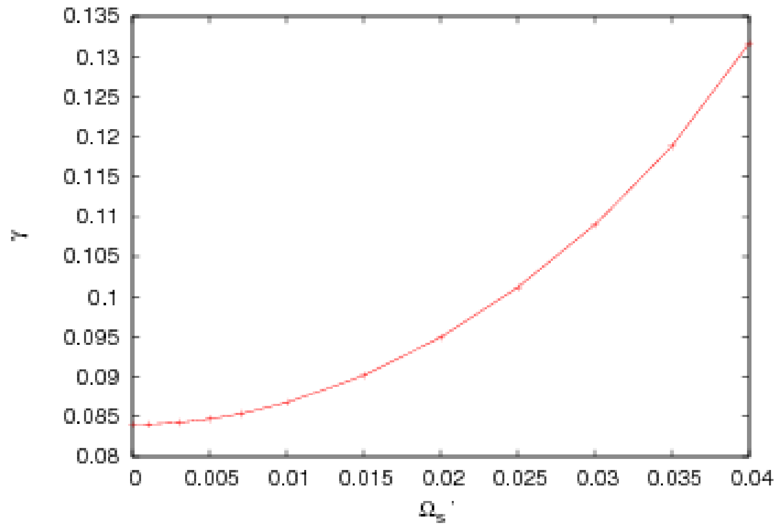
Numerical solution to the inner-layer equation

- The inner-layer equations are solved as an eigenvalue problem under boundary conditions of the third kind:

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}X} = \Phi_0 \dfrac{\psi}{c+X} \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}X} = \dfrac{\psi}{c+X} \end{cases}$$

(Φ_0 has been already determined by the asymptotic analysis of inner-layer equation)

- An example for $c = 5$; c is inversely proportional to Δ', to be given by external solution
- The flow velocity is sub-Alfvenic



(Limit approaching zero flow must be understood carefully)

- Increase of the growth rate due to the shear flow is observed

Equilibrium for global calculation in slab geometry

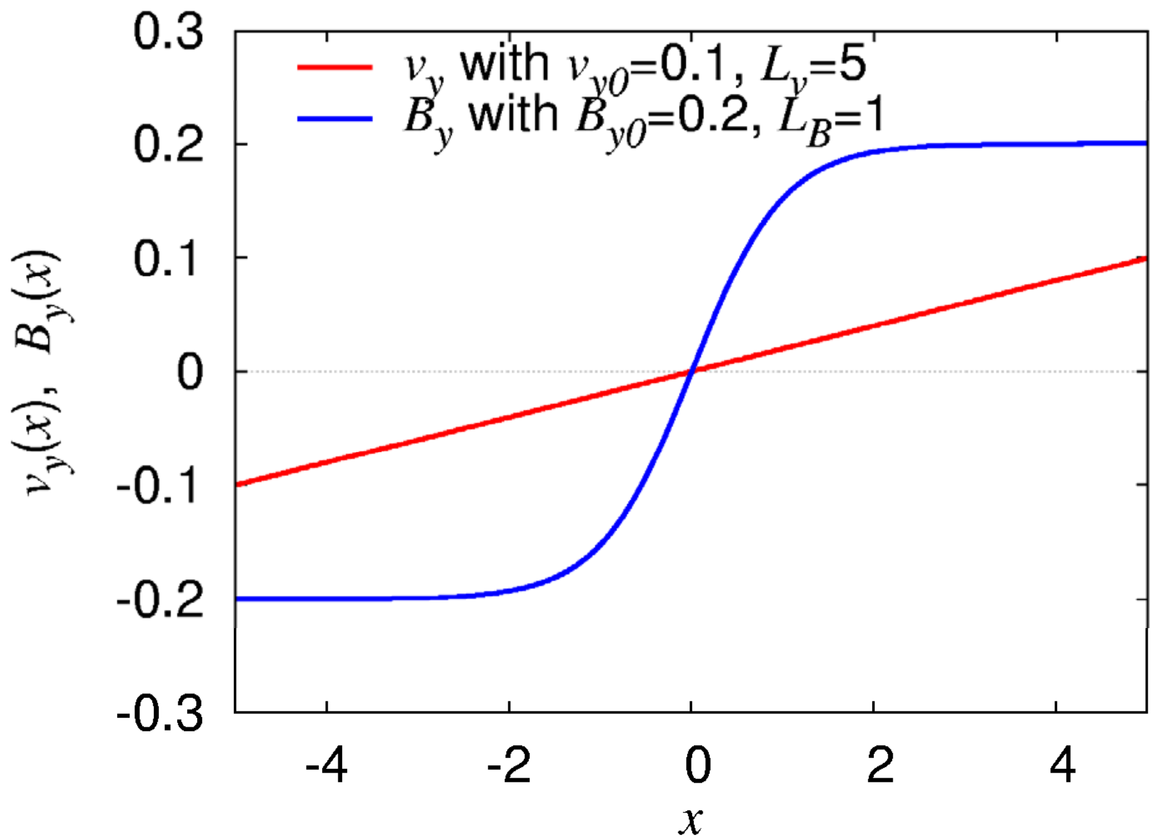
- Velocity and magnetic fields

$$v_y(x) = v_{y0}\frac{x}{L_v}$$

Stable to Kelvin-Helmholtz modes without magnetic field

$$B_y(x) = B_{y0}\tanh\frac{x}{L_B}$$

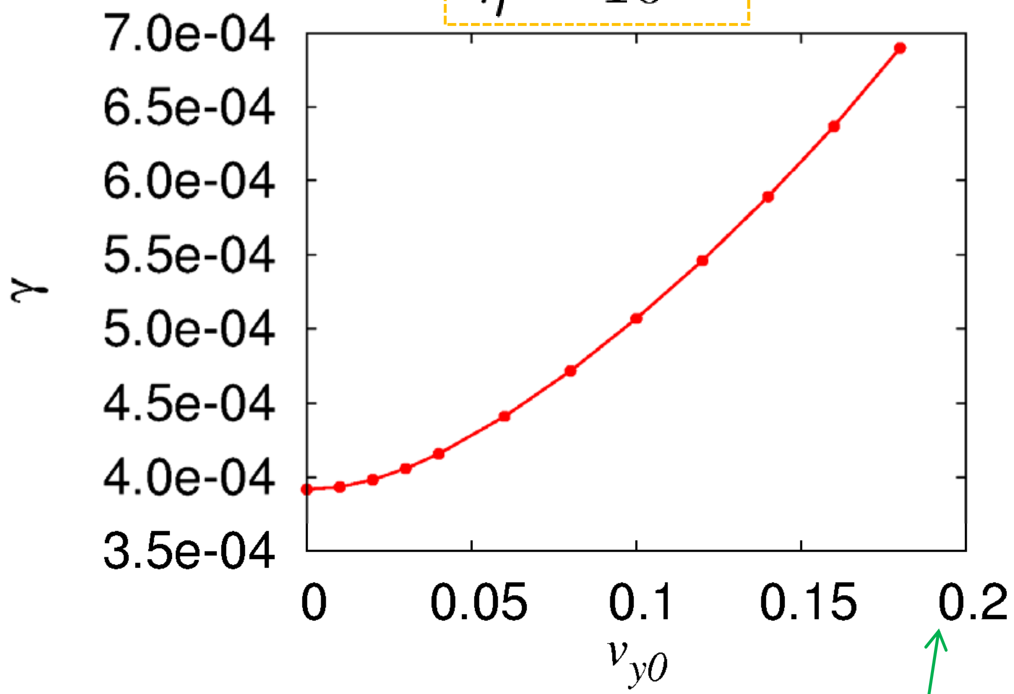
Tearing mode is unstable for $k_y < 1$ without flow

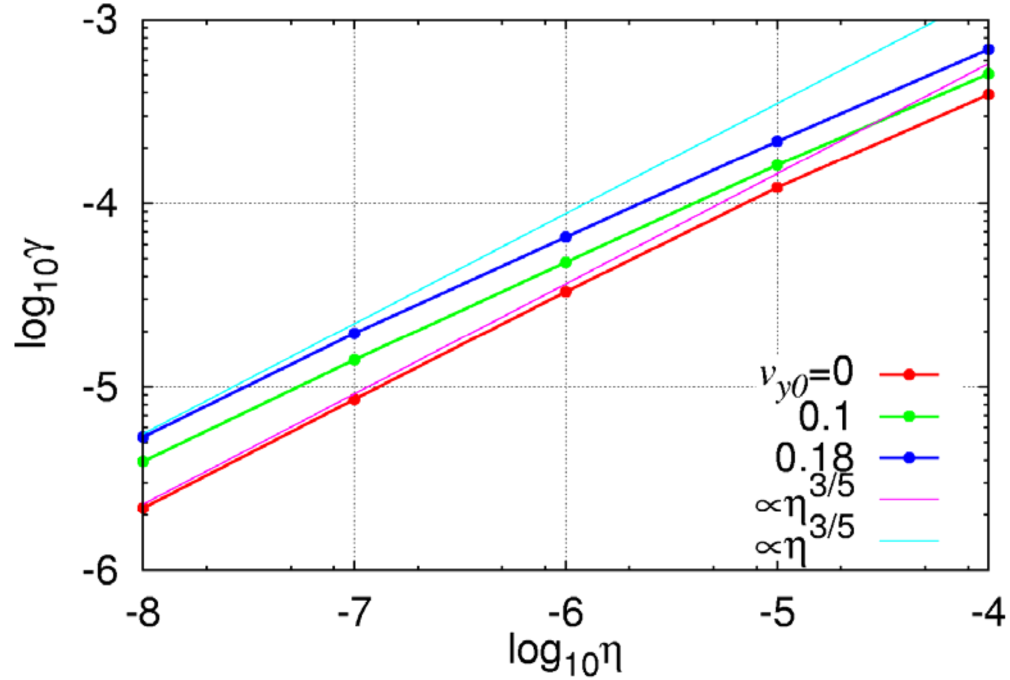


Eigenvalue

$k_y = 0.9$

$\eta = 10^{-4}$





Sub-Alfvenic flow
in the whole domain

- Growth rate icreaeses as the flow is increased
- It scales as $\gamma \propto \eta^{3/5}$ for small η

Summary

- Asymptotic matching is possible in the presence of sheared plasma rotation:
  - The rotation is assumed to be zero at the mode-resonant surface
    - We can assume this without loss of generality since the finite rotation speed at the resonant surface just introduces a finite frequency of the mode under fixed boundary condition
  - Outer solution can be expressed by a Frobenius series since the mode-resonant surface remains as a regular singularity even if a plasma rotation is included
  - Asymptotic series solution to the inner-layer equation is shown to match onto the outer solution
    - The convection term as well as the line-bending term become dominant away from the mode-resonant surface

- The inner-layer equation was solved numerically under the boundary condition of the third kind, which was derived from the asymptotic form of the inner-layer solution

- It was shown that the growth rate of the tearing mode becomes larger by increasing the sheared rotation