Adaptive Stochastic OPTIMAL Control Via Output Feedback

(FOR RWM)

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Motivation and Outline

- Need for advanced modern control system.
- Optimal state feedback (in noise)
  - Too complex,
  - Not user friendly,
- Optimal output feedback (in noise)
  - Less complex,
  - Better performance
- Adaptive optimal output feedback (in noise)
  - Adaptation is a must in time evolving plasma discharge,
  - Identification of evolving plasma instability parameters,
- Evolving controller design based on the above

OPTIMAL STATE FEEDBACK (IN NOISE)

Basic Equations of a Single RWM

- Variables: I_1 (plasma current), I_2 (wall current), I_3 (control current).

$$L_1^{eff} I_1 + M_{12} I_2 + M_{13} I_3 = \psi_n \,(\text{state noise})$$
$$\gamma M_{12} I_1 + (\gamma + \tau_2^{-1}) L_2 I_2 + \gamma M_{23} I_3 = 0$$
$$\gamma M_{13} I_1 + \gamma M_{23} I_2 + (\gamma + \tau_3^{-1}) L_3 I_3 = u(\text{input})$$

- The above equations can be generalized:

$$\dot{I} = AI + Bu + D\psi_n \qquad (1)$$
$$\psi(t) = H^T I(t) + \psi_m(t)\,(measurement \quad noise) \qquad (2)$$

- Goal: minimize fluctuation energy and control energy, i.e., minimize

$$J = \frac{1}{T_f} \int_0^{T_f} E[I^T(t) Q I(t) + u^T(t) R u(t)]dt, T_f \to \infty \qquad (3)$$

subject to the constraint of Eq. (1).

• Use calculus of variations: defining a Lagrangian L

$$L(x,\dot{x}) = I^T(t)QI(t) + u^T(t)Ru(t) + \lambda^T(t)[AI(t) + Bu(t) - \dot{I}(t)]$$

where $x = (I^T\ U^T\ \lambda^T)^T$, λ is a Lagrange multiplier.

• Then the optimal control minimizing J of Eq. (3) satisfies the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

One resulting equation is:

$$u(t) = -R^{-1}B^T\lambda(t) \qquad (4)$$

Two other resulting equations:

$$\begin{pmatrix} \dot{I}(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} I(t) \\ \lambda(t) \end{pmatrix}$$

- **It can be shown that** $\lambda = SI$, **where** S is the solution of the matrix Riccati Eq:

$\boxed{SA + A^T S - SBR^{-1}B^T S = -Q}$. Then from Eq. (4):

$$\text{STATE FEEDBACK } \boxed{u(t) = -K_c I(t)} = -R^{-1}B^T SI(t) \qquad (5)\ ;\ K_c = R^{-1}B^T S$$

Note: optimal feedback is necessarily stabilizing!

- **Solution**: assume $\psi_n(t)$ is white noise, $\psi^2_{nRMS} = W$:

$$(A - BK_c)I^2_{RMS} + I^2_{RMS}(A - BK_c)^T = -DWD^T$$

Design of a State Observer (Kalman Filter)

- **Sensor output:** $\boxed{\psi(t)} = \underbrace{HI(t)}_{\text{Linear combination of states}} + \psi_m(t)\,(measurement \quad noise)$

- **Observer Eq:** $\dot{\hat{I}}(t) = A\hat{I}(t) + Bu(t) + K_f[\psi(t) - H\hat{I}(t)]$

- **Determination of K_f:** estimation error $e = I - \hat{I}$:

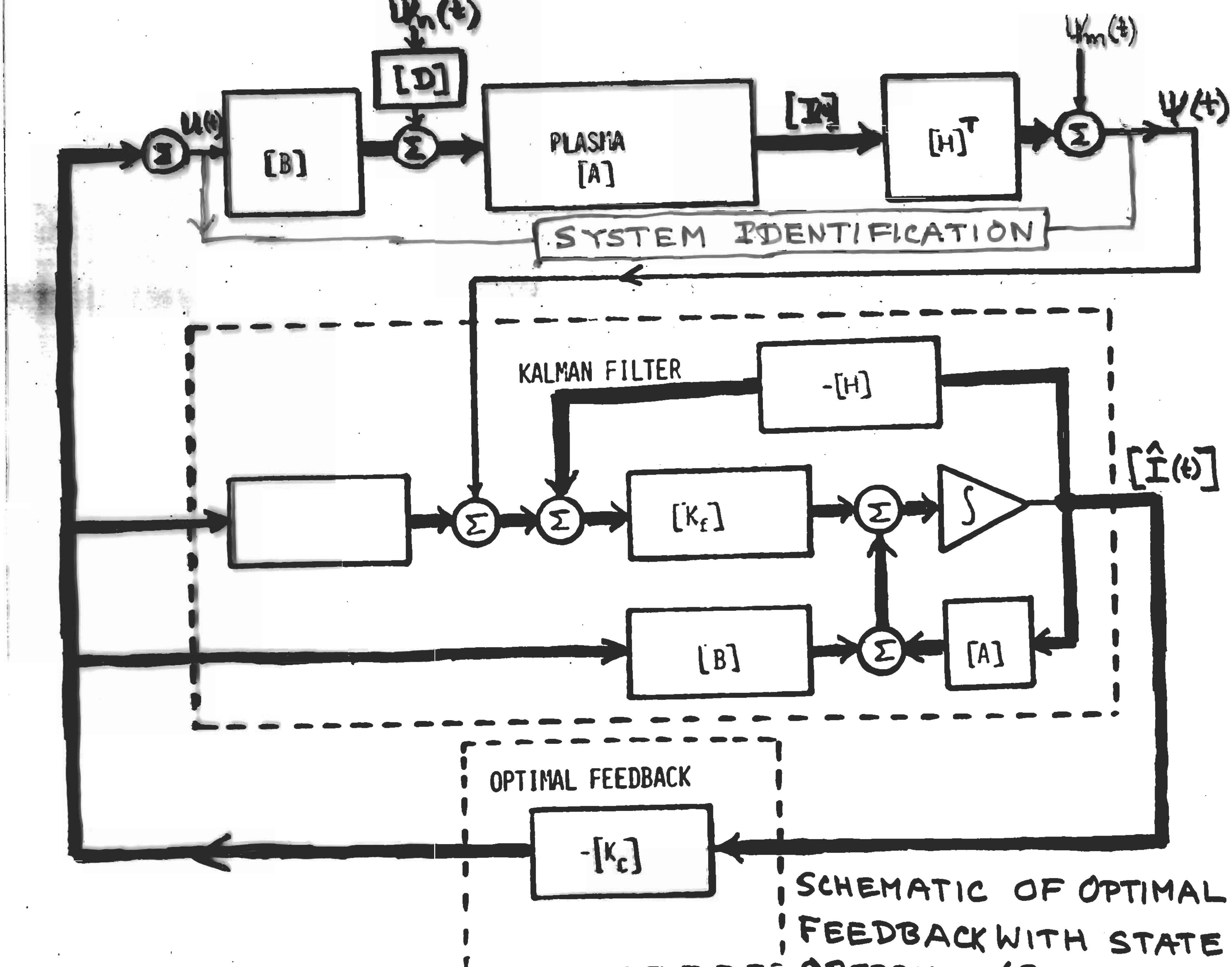
$$\dot{e}(t) = (A - K_f H)e(t) + D\psi_n(t) - K_f \psi_m(t)$$

- Minimization of $P(t) = E[e(t)e^T(t)]$ via a similar procedure of variational calculus to yield the observer Ricatti Eq:

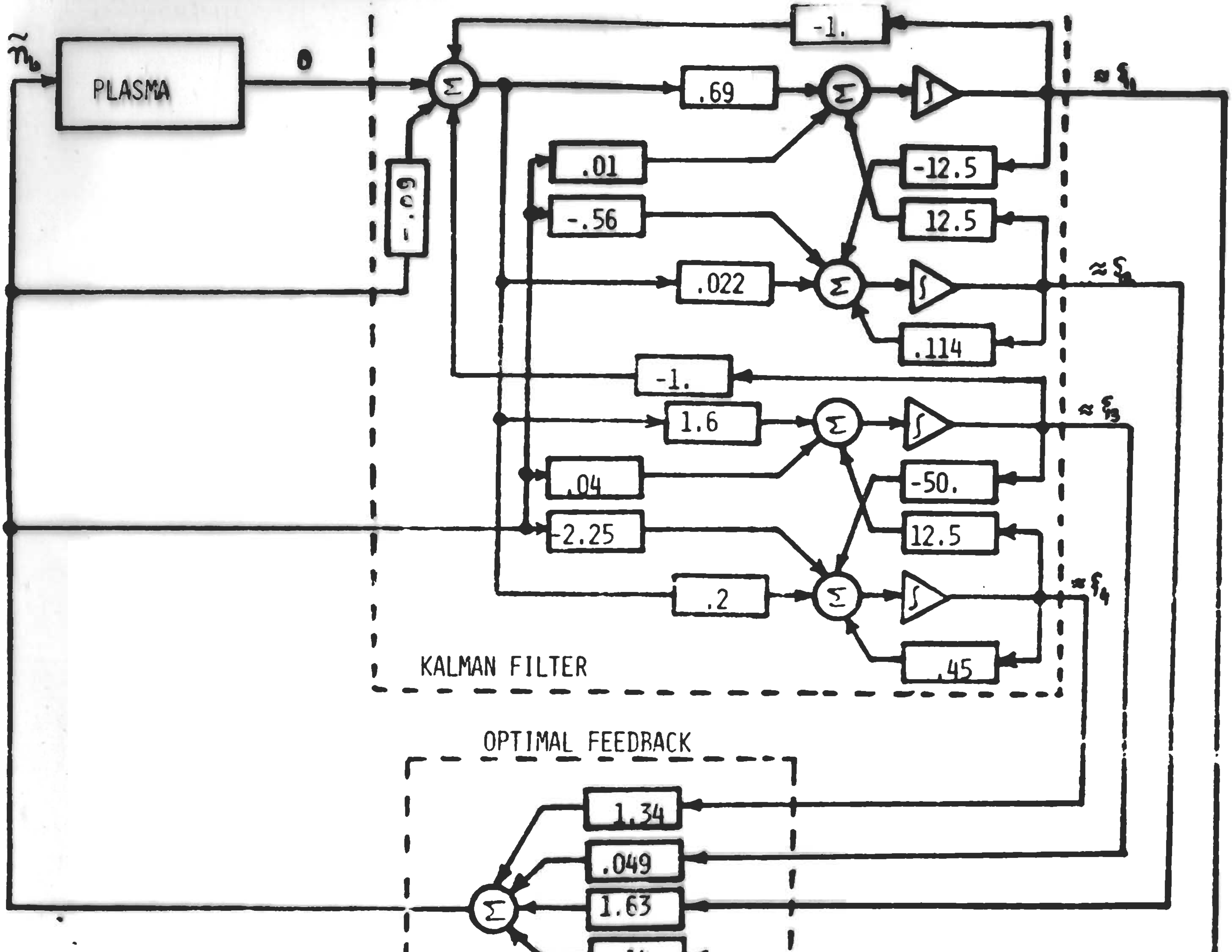
$$0 = (A - K_f H)\bar{P} + \bar{P}(A - K_f H)^T + DWD^T + K_f V K_f^T$$

$$K_f = \bar{P}H^T V^{-1}; \psi_{nRMS}^2 = V$$

Then $\quad I_{RMS}^2 = \hat{I}_{RMS}^2 + \bar{P}, E[uu^T] = K_c \hat{I}_{RMS}^2 K_c^T$



SCHEMATIC OF OPTIMAL FEEDBACK WITH STATE OBSERVER (RECONSTRUCTOR)



• DETAILED FEEDBACK DESIGN

TRANSFER FUNCTION MODEL OF

Basic Equations of a Single RWM

- The continuous system model in transfer function form:

$$A(s)\psi(s) = B(s)u(s) + C(s)e(s) \qquad (1)$$

  - $A(s) = s^2 - 78.5s - 7.4\times10^3$, the poles are [-55, 133],
  - $B(s) = -1.6s + 541.1$,
  - $C(s) = s^2 + 4.5\times10^3 s - 5.9\times10^6$
  - The term $e(s)$ is the system noise, including both state noise ψ_n and measurement noise ψ_m.
- The sampling rate of the system model is chosen to be 1ms and the resulting discrete transfer function is: (q is the forward shift operator, i.e., $q\psi(k) = \psi(k+1)$)

$$A(q)\psi(k) = B(q)u(k) + C(q)e(k) \qquad (2)$$

  - $A(q) = q^2 + a_1 q + a_2 = q^2 - 2.1q + 1.1$, $B(q) = b_0 q + b_1 = (-1.37q + 1.94)\times10^{-3}$
  - $C(q) = q^2 + c_1 q + c_2 = q^2 - 0.36q - 6.74 = (q - 2.78)(q + 2.42)$

- The optimal output feedback controller requires that $C(q)$ has all its zeros inside the unit disc.
  - If $C(q)$ has zeros outside unit circle, factorize $C = C^+C^-$,

    where C^- contains all factors with zeros outside the unit circle.
  - Replace C^- with its reciprocal form C^{-*}.
  - $C(q) = (q+1/2.42)(q-1/2.78) = q^2+0.05q-0.15$
- The broad band RMS noise is roughly 1/2 to 1 Gauss
  - It is considered that the magnitude of the plant noise is a fraction of that of the measurement noise, so we assume

    $\psi_n = 3\times10^{-6}$ Weber, $\psi_m = 2\times10^{-4}$ Weber.

OPTIMAL OUTPUT FEEDBACK

- The main goal is to minimize both the fluctuation energy of the instabilities and the control energy simultaneously, so the quadratic cost function is:

$$J = E\{(\underbrace{\psi(k)}_{\text{output flux}})^2 + \overbrace{\rho}^{\text{weighting}}\underbrace{u}_{\text{control input}}^2\} \tag{3}$$

where ρ is the relative weight of control energy over fluctuation energy.

- Assumptions:
  - $C(q)$ has all its zeros inside the unit disc,
  - There is no polynomial which divides $A(q)$, $B(q)$ and $C(q)$.

Using a similar formalism of calculus of variations one can find the admissible control law which minimizes (3) with $\rho > 0$ is given by:

$$R(q)u(q) = -S(q)\psi(q) \tag{4}$$

- $R(q)$ and $S(q)$ satisfy the Diophantine equation:

$$A(q)R(q)+B(q)S(q)=P(q)C(q) \tag{5}$$

- The polynomial $P(q)$ is the solution of the spectral factorization problem:

$$rP(q)P(q^{-1})=\rho A(q)A(q^{-1})+B(q)B(q^{-1}) \tag{6}$$

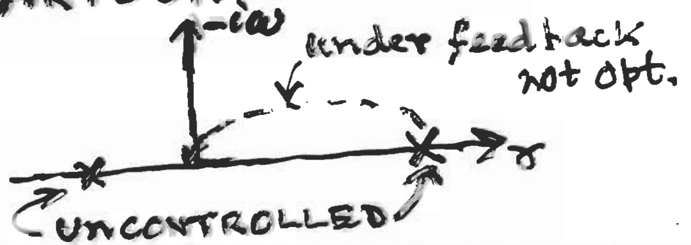
where r is a coefficient that can be uniquely solved from the above equation.

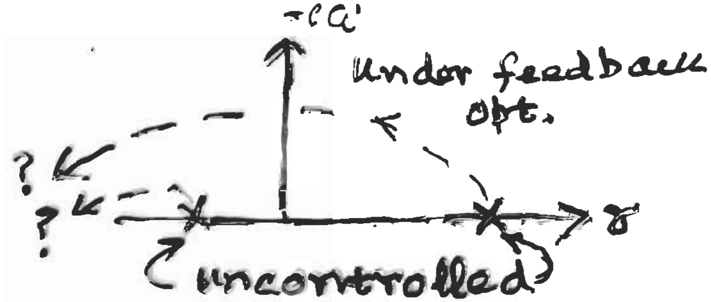
> ➢ $P(q)$ gives the closed loop pole of the system.
>
> ➢ Eq.(6) can be solved directly or iteratively.

- The optimal output feedback controller is closely connected to the pole placement method.

> ➢ The solution of the Diophantine equation Eq. (5) can be interpreted as a pole placement problem.

CARTOON OF CONCEPT:





Online System Identification

Batch Least Square Method Based on a Deterministic Model

- The deterministic model is used to derive the Batch LS method.

$$A(q)\psi(k) = B(q)u(k) \tag{7}$$

Output Input

- Apply a sequence of inputs $\{u(1)\ldots u(k)\ldots u(n)\}$ to the plasma system and a sequence of outputs $\{\psi(1)\ldots\psi(k)\ldots\psi(n)\}$ is obtained.

- Define the parameter vector $\theta^T = \begin{pmatrix} a_1 & a_2 & b_0 & b_1 \end{pmatrix}$ and the regression vector $\varphi^T(k-1) = \begin{pmatrix} -\psi(k-1) & -\psi(k-2) & u(k-1) & u(k-2) \end{pmatrix}$, the input-output relation is:

$$\psi(k) = \varphi^T(k-1)\theta \tag{8}$$

Scalar output Regression vector Parameter vector

- **Define :** $\Phi = \begin{pmatrix} \vdots \\ \varphi^T(k-1) \\ \vdots \\ \varphi^T(n-1) \end{pmatrix}$ $\quad \Psi = \begin{pmatrix} \vdots \\ \psi(k) \\ \vdots \\ \psi(n) \end{pmatrix}$, Eq. (8) becomes: $\Psi = \Phi\theta$.

Output vector Regression matrix (square) Parameter vector

- **The objective is to determine** the parameter vector $\hat{\theta}$ in such a way that the **computed outputs agrees** with the output as close as possible in the sense of least **square.**

  - ➢ $\hat{\theta}$ **is the estimate** of θ.
  - ➢ **The least square** loss function is defined as:

$$V(\theta,n) = \frac{1}{2}\sum_{k=1}^{n}\left(\psi(k) - \varphi^T(k)\hat{\theta}\right)^2 \tag{9}$$

  - ➢ **If the matrix** $\Phi^T\Phi$ is nonsingular, $\hat{\theta}$ is unique and given by

$$\boxed{\hat{\theta} = \left(\Phi^T\Phi\right)^{-1}\Phi^T\Psi} \tag{10}$$

Recursive Least Square (RLS) Method

- It is desirable to compute the estimate recursively. First, define the covariance

matrix P: $P(k)=\left(\Phi^{T}(k)\Phi(k)\right)^{-1}=\left(\sum_{i=1}^{k}\varphi(i)\varphi^{T}(i)\right)^{-1}$.

- Eq.(10) can be rewritten as: $\hat{\theta}(k)=P(k)\left(\sum_{i=1}^{k}\varphi(i)\psi(i)\right)=P(k)\left(\sum_{i=1}^{k-1}\varphi(i)\psi(i)+\varphi(k)\psi(k)\right)$.

- Use the definition of P, the following equation is obtained:

$$\sum_{i=1}^{k-1}\varphi(i)\psi(i)=P^{-1}(k-1)\hat{\theta}(k-1)=P^{-1}(k)\hat{\theta}(k-1)-\varphi(k)\varphi^{T}(k)\hat{\theta}(k-1).$$

- The recursive least square (RLS) method takes the form:

$$\hat{\theta}(k)=\hat{\theta}(k-1)+\overbrace{K(k)}^{\text{weighting}}\overbrace{(\psi(k)-\varphi^{T}(k)\hat{\theta}(k-1))}^{\text{correction}}$$

$$K(k)=P(k)\varphi(k)=P(k-1)\varphi(k)(I+\varphi^{T}(k)P(k-1)\varphi(k))^{-1} \qquad (11)$$

$$P(k)=(I-K(k)\varphi^{T}(k))P(k-1)$$

Extended Least Square (ELS) Method

- The stochastic model Eq. (2) is used to derive the ELS method.
- For a stochastic system, the RLS method Eq. (11) can not be used directly because the regression vector and the disturbances are correlated, i.e., $E\left[\varphi^T e\right] \neq 0$.
- Introduce: $\varepsilon(k) = \psi(k) - \varphi^T(k-1)\hat{\theta}(k-1)$ to estimate the noise term $e(k)$, $\theta = \left(a_1\ a_2\ b_0\ b_1\ c_1\ c_2\right)$, $\varphi^T(k) = \left(-\psi(k)\ -\psi(k-1)\ u(k)\ u(k-1)\ \varepsilon(k)\ \varepsilon(k-1)\right)$, then the RLS method can be used. This method is called ELS method.
- The identified system model is shown in Fig.1.
  - The convergence time of the system identification is 10ms.
  - The value of initial P matrix determines the convergence time.
- The growth rate of the open loop system is shown in Fig.2(a).

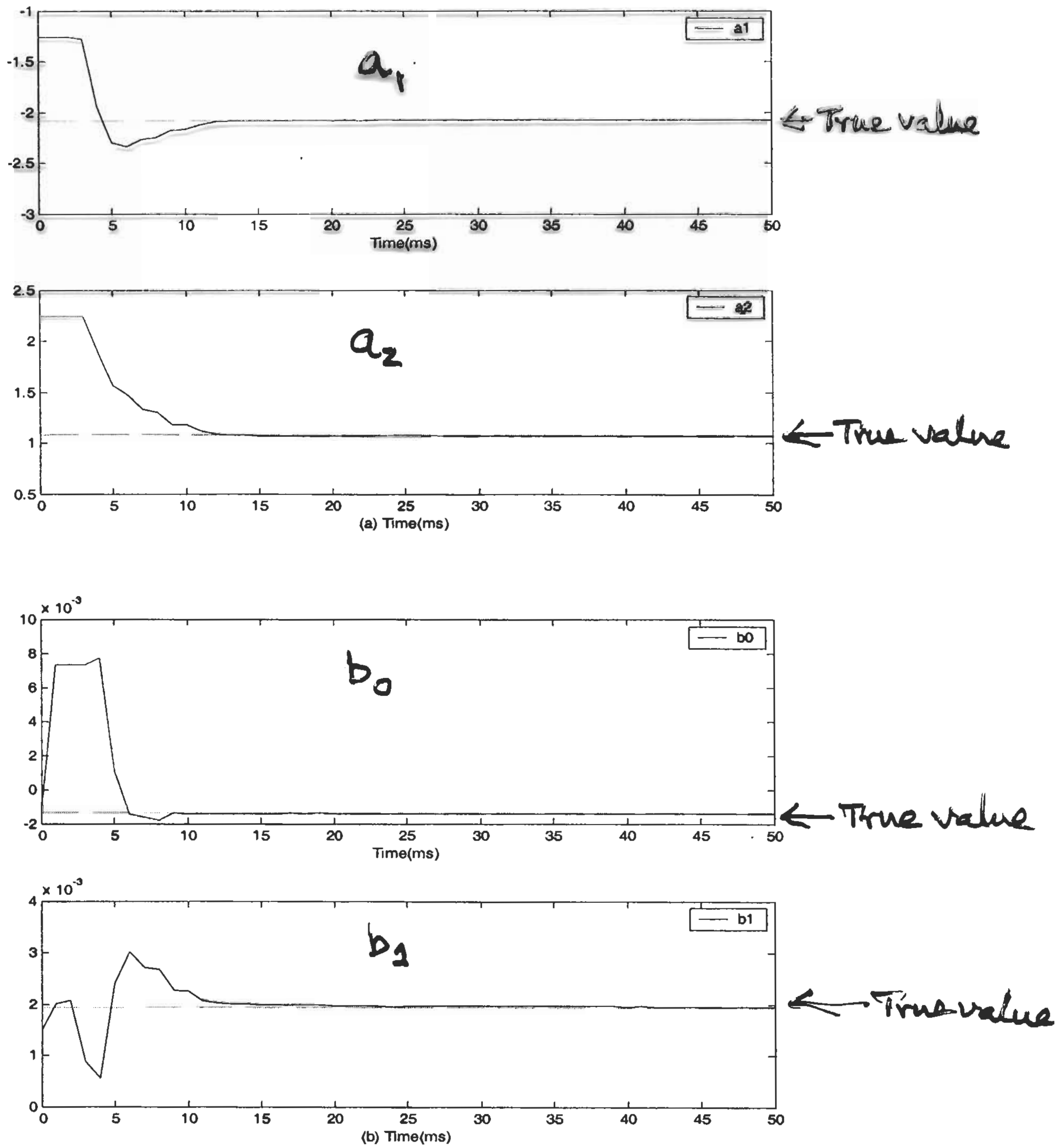


Fig. 1 Identification of a 'time invariant' system with ELS method. The solid lines are the estimate and the dashed lines are true values.

Convergence time ~10 ms

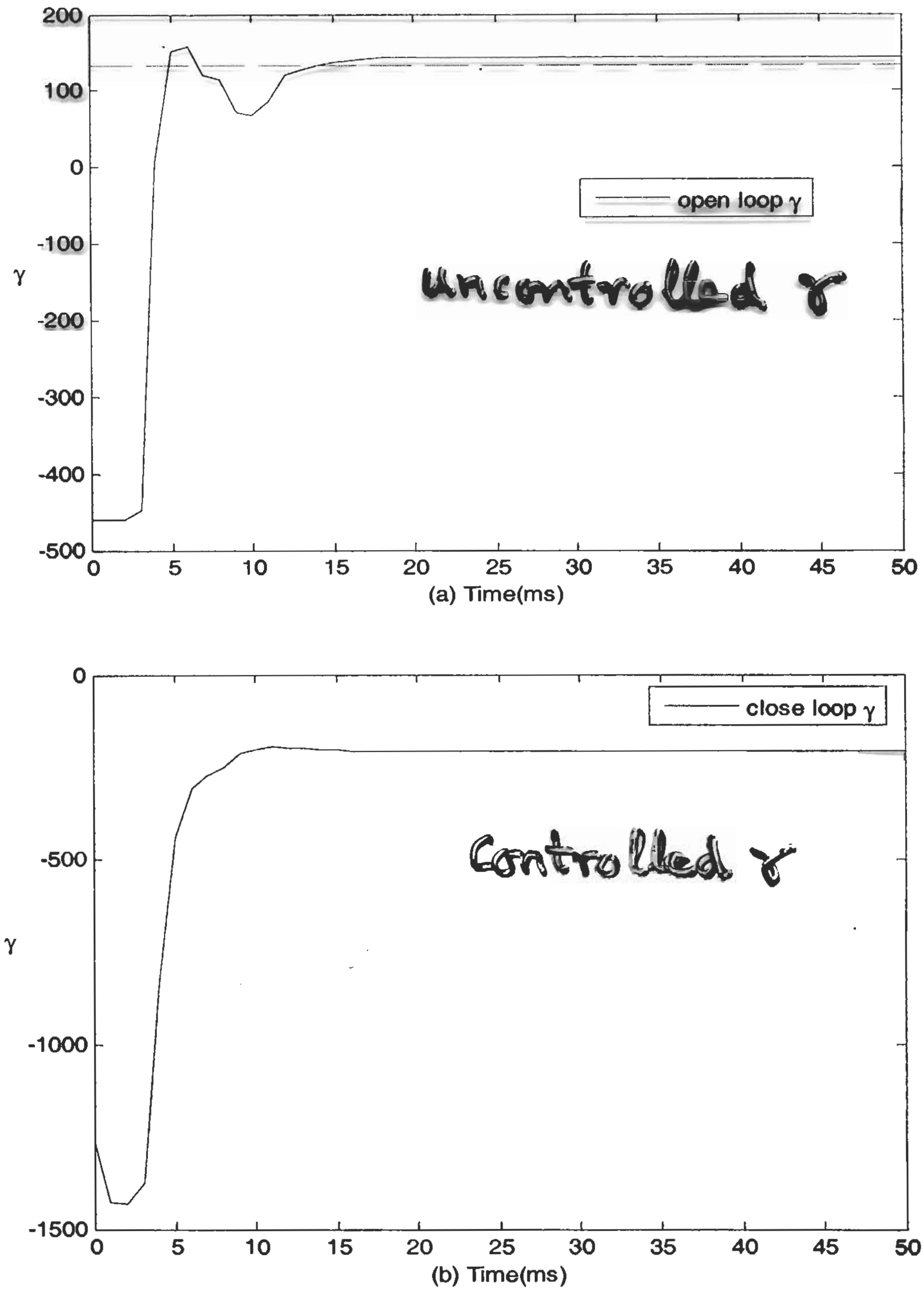


Fig. 2 Growth rate. (a) is the growth rate of the open loop system and (b) is the closed loop system. Negative growth rate means the system is stable.

Convergence time ~ 10 ms

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Optimal Control of the Identified Model

Optimal Control of the Time Invariant System

- A simulation of the optimally controlled time invariant plasma system is shown.
  - Fig.2(b) is the damping rate of the close loop time invariant system.
  - Fig.3(a) is the estimated controlled plasma current. The RMS value is 420A×3
  - Fig.3(b) is the control current. The RMS value is 36A. ×3
  - The control response time is 20ms.
    - The identification takes 10ms to converge, so the total 30ms.
  - The controller stabilizes when the system identification converges.
  - ρ has very big effect on the convergence time. A larger ρ will increase the convergence time.

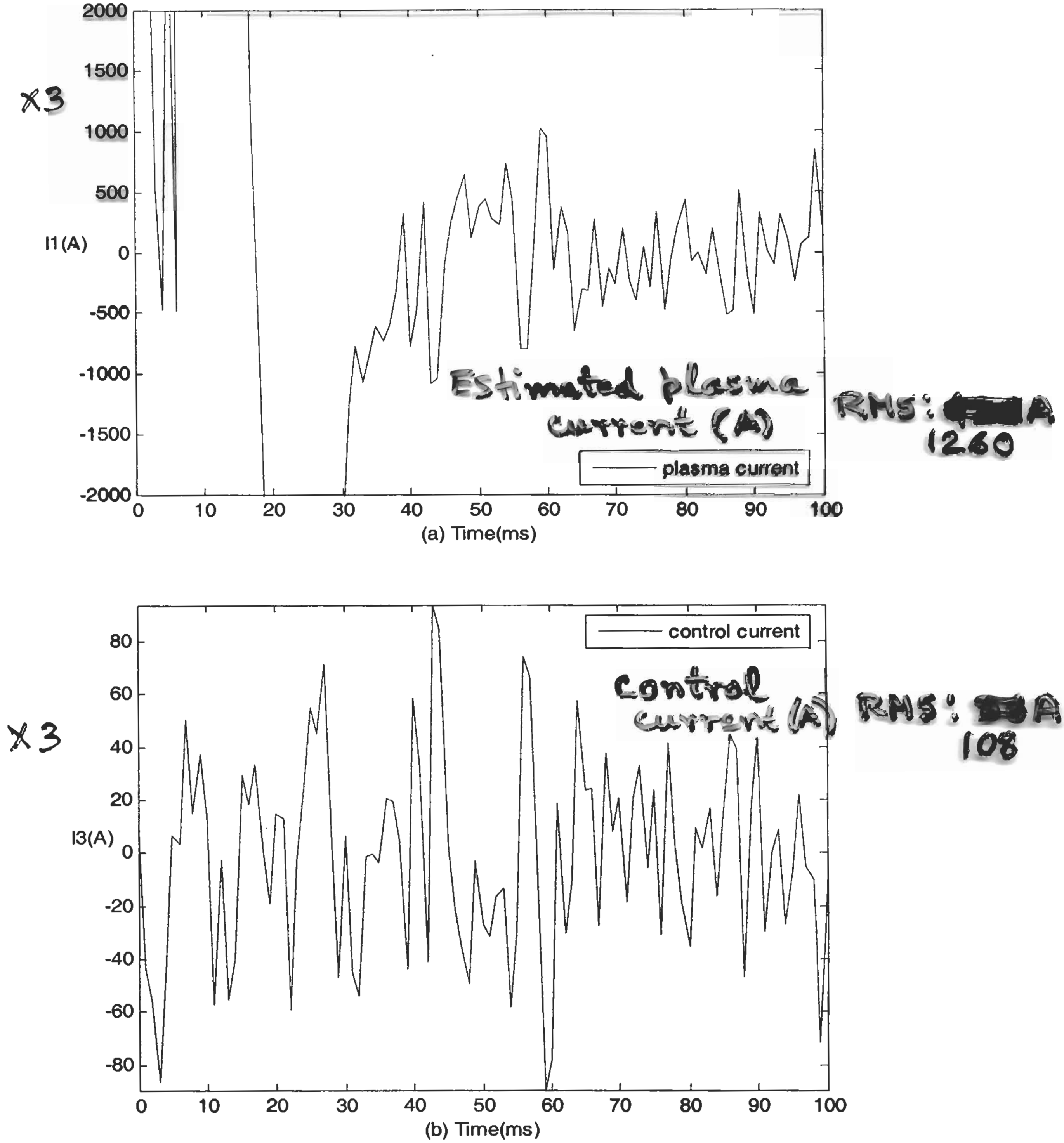


Fig. 3 [redacted] optimal control of a "time invariant" system.

Extended Least Square (ELS) Method with Forgetting Factor

- The real plasma systems are always dynamic and evolving.
- One method to estimate the slowly time-evolving system parameters is to use a forgetting factor λ, $0 < \lambda \leq 1$, in the identification.
- The ELS method with a forgetting factor becomes:

$$\begin{aligned}
\hat{\theta}(k) &= \hat{\theta}(k-1) + K(k)(\psi(k) - \varphi^T(k)\hat{\theta}(k-1)) \\
K(k) &= P(k)\varphi(k) = P(k-1)\varphi(k)(\lambda I + \varphi^T(k)P(k-1)\varphi(k))^{-1} \\
P(k) &= (I - K(k)\varphi^T(k))P(k-1)/\lambda
\end{aligned} \tag{12}$$

- The relationship between the forgetting factor λ and the time constant of this method, T_f, is: $\lambda = e^{-T_s/T_f}$
  - ➢ fast evolution → quick discount of the old data → smaller λ.
  - ➢ slow evolution → slow discount of the old data → larger λ.

- Simulation of a time evolving system. The simulation starts with the original system, then the poles of the open loop system is increased by 10% of the original value after 50ms and this is repeated ten times. The final poles are two times the original value.

$$(-55\ 133) \rightarrow (-55\ 133)\times 1.1 \rightarrow (-55\ 133)\times 1.2 \ldots \rightarrow (-55\ 133)\times 2$$

- The identified system model is shown in Fig.4.
  - The estimator follows the evolution of the system closely. That means this identification algorithm can be used in an adaptive controller.
- The growth rate of the open loop system is shown in Fig.5(a). The oscillation of the growth rate is caused by the change in the system model.

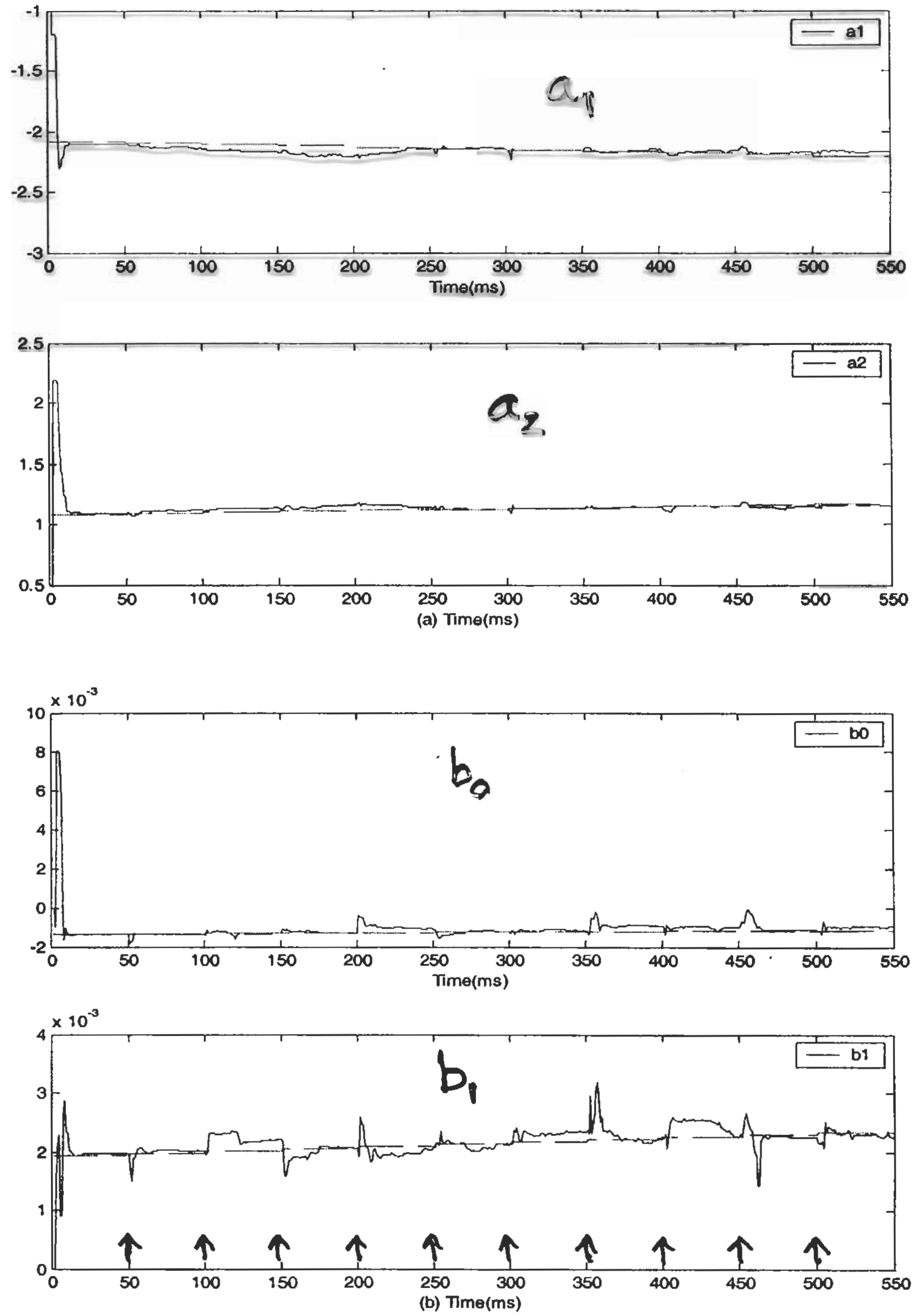


Fig.4 Identification of a time evolving system with forgetting ELS. The solid lines are estimates and the dashed lines are true values.

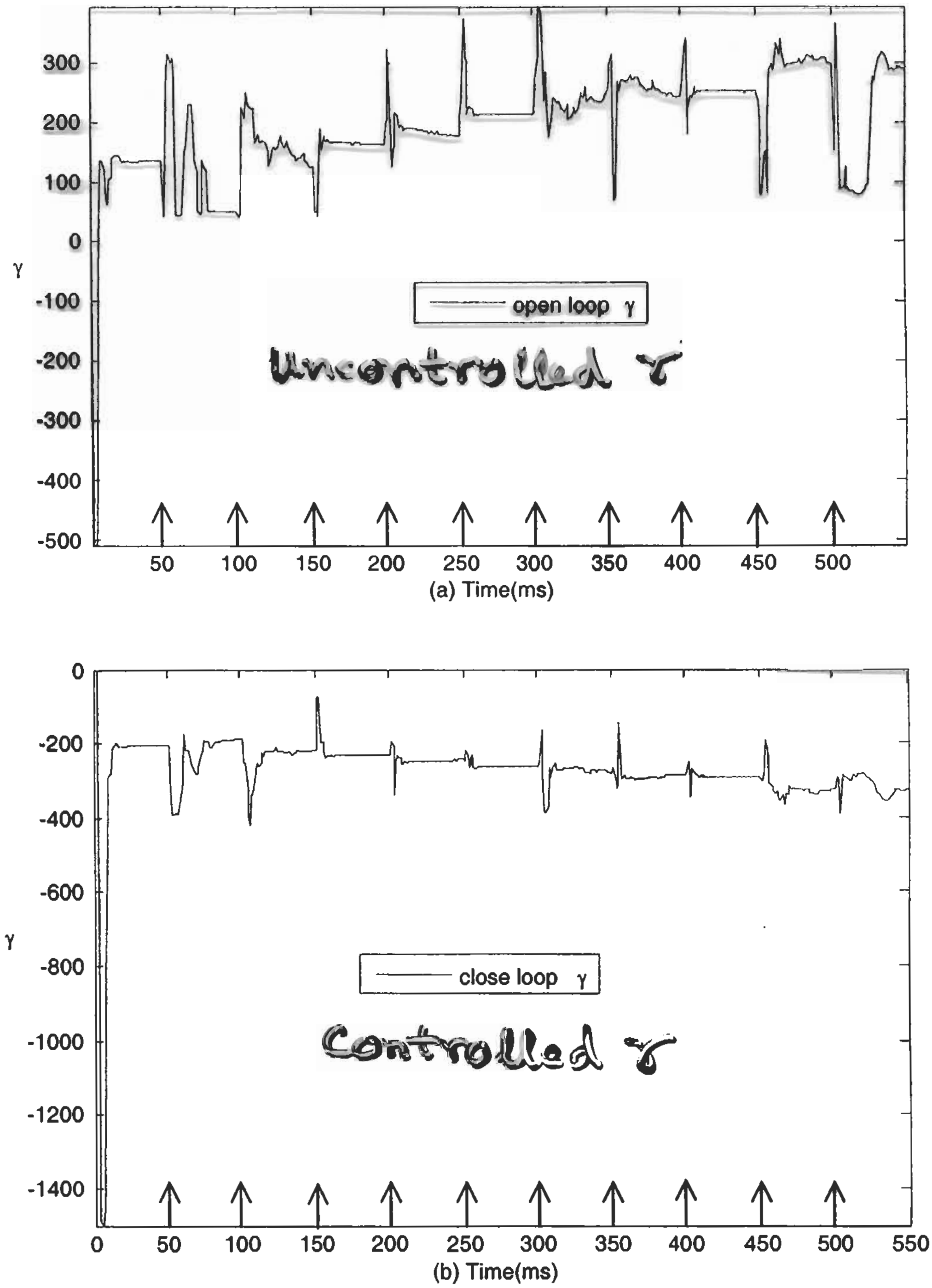
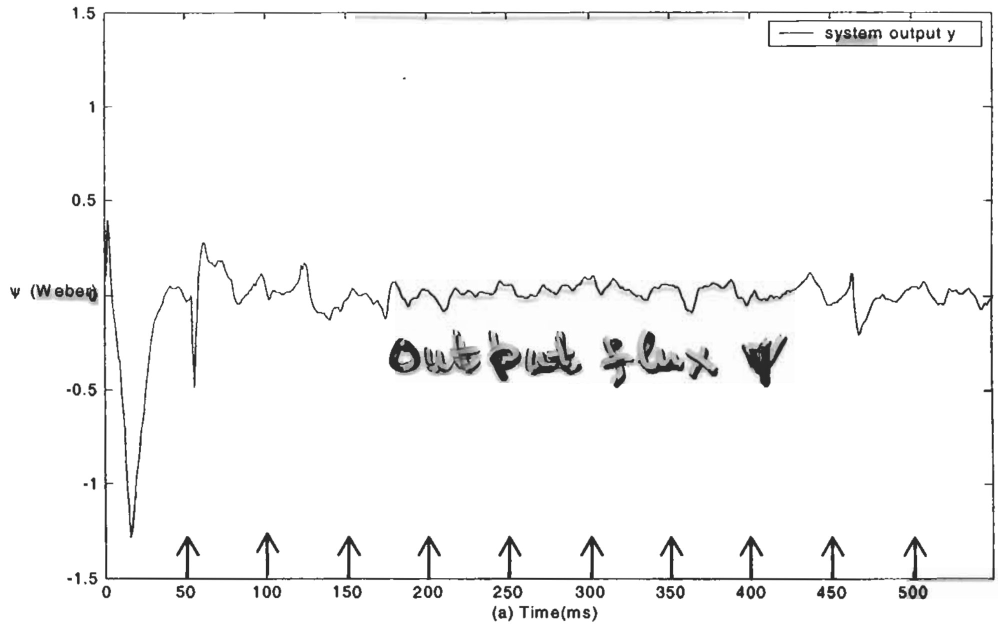


Fig. 5 Growth rate. (a) is the growth rate of the open loop system and (b) is the closed loop system. The arrows indicate where the system change takes place.

Optimal Control of the Time evolving System

- A simulation of the optimally controlled time evolving plasma system is shown.
  - Fig.5(b) is the damping rate of the close loop time evolving system.
  - Fig.6(a) is the system output measurement.
  - Fig.6(b) is the control signal.



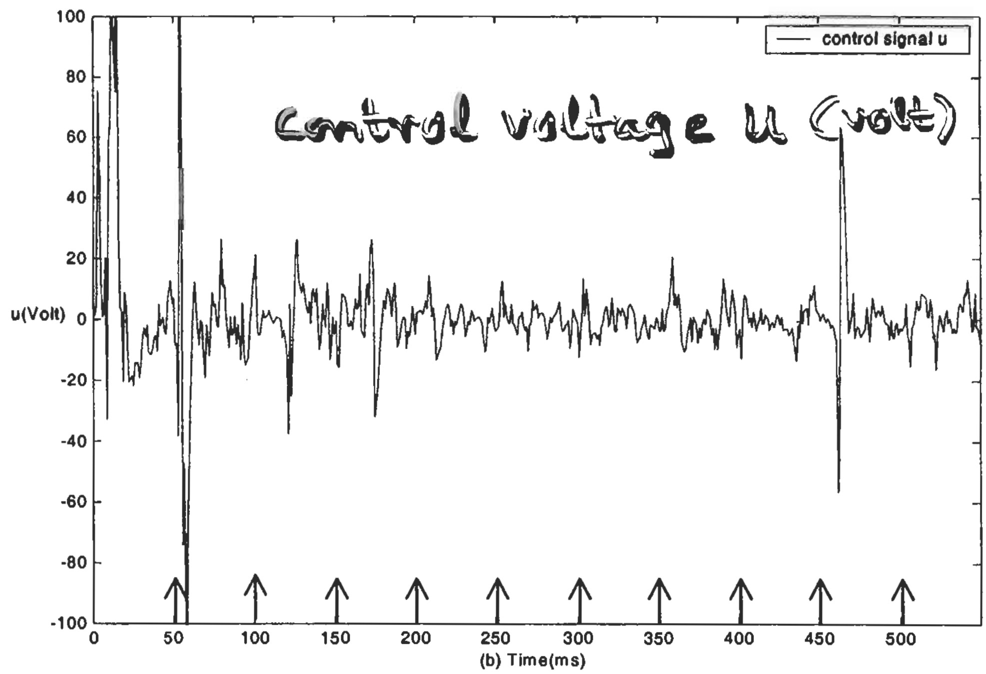


Fig. 6 Adaptive control of a time evovling system. The arrows indicate where the system change takes place.

Conclusions

• Plasma noise and measurement noise modeling is ~~somewhat~~ quite questionable,

• Compared with the stochastic optimal state feedback control studied before, the optimal output feedback controller is better in some aspects:

- ➢ The implementation is simplified,
  - ❖ The estimation of the system states is unnecessary and the Kalman filter is not needed,
  - ❖ The optimal control design is simplified,
- ➢ Therefore, the system identification and control response time are shorter,
- ➢ The system identification is more accurate,

• **Adaptive optimal control appears to be feasible for slow growing modes like RWMs. It is a must for the future magnetic fusion machines.**

• **In principle, all plasma instabilities with discrete spectra can be feedback stabilized (observability and controllability): demonstrated theoretically and experimentally in CLM.**