Mechanism of stabilization of ballooning modes by toroidal rotation shear in tokamaks and analytic formula for stability estimatin

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Outline

- **First part:**
  - **Background and Motivation**
  - **Physical mechanism of ballooning-mode stabilization by toroidal rotaion shear is clarified:**
    Energy transfer from an unstable mode to countably infinite number of stable modes
- **Second part:**
  - **Background and Motivation (II)**
  - **Semi-analytic formula is derived for roughly estimating the toroidal rotation shear required to stabilize the ballooning mode**
- **Conclusions**

Background

- **High-n (toroidal mode number) ballooning modes in toroidally rotating tokamaks have been theoretically studied by using time-dependent eikonal representation**

  [F. L. Waelbroeck and L. Chen, Phys. Fluids B **3**, 601 (1991).]

- **The resultant ballooning equations are two coupled wave equations along a magnetic field line**

- **The wave vector, which is defined by a gradient of the eikonal, depends on time as**

  $\mathbf{k} \equiv \nabla\zeta - q\nabla\theta - (\vartheta - \tilde{\theta}_k(t))\nabla q$ $\tilde{\theta}_k(t) \equiv \theta_k - \dot{\Omega}t$ $\dot{\Omega} \equiv \mathrm{d}\Omega/\mathrm{d}q$

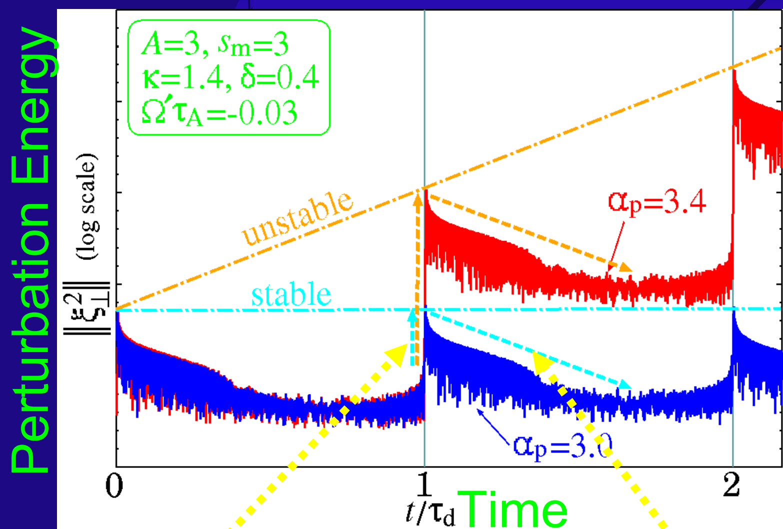
  that is, the ballooning angle changes in time effectively

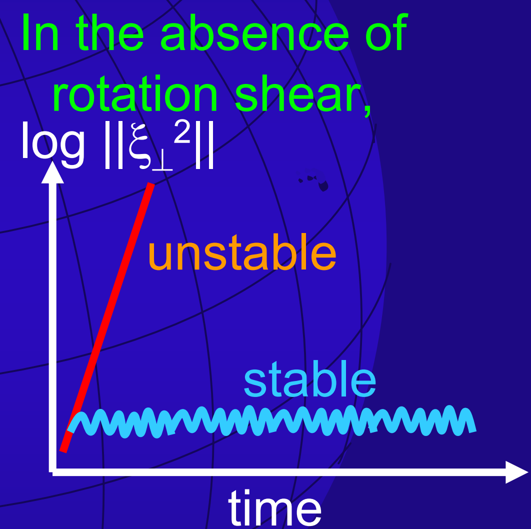
- **The coefficients of the ballooning equations are time dependent throught the wave vector, thus we have solved them as an initial-value problem**

Motivation

- **We found that toroidal rotation shear causes damping of perturbation energy of high-n ballooning mode, which is the stabilization mechanism by the rotation shear**

  [M. Furukawa, S. Tokuda and M. Wakatani, Nucl. Fusion **24**, 1579 (2003).]





Exponential growth (bad curvature) < Damping (good curvature) → Stable

- **The physical mechanism how the damping occurs has not been clarified**

Model equation for ballooning modes in a toroidally rotating tokamak

[M. Furukawa, S. Tokuda, 13th Int. Toki Conf. (Toki, Japan, 2003).]

- **We have derived a model equation for ballooning modes in a toroidally rotating tokamak as**

$$\bar{\rho}\left(\frac{\partial^2 \xi_\perp}{\partial t^2} - U\frac{\partial \xi_\perp}{\partial t}\right) = \mathcal{L}\xi_\perp$$

:A wave equation along a magnetic field line

$$\mathcal{L}\xi_\perp \equiv \frac{\partial}{\partial \vartheta}\left(f\frac{\partial \xi_\perp}{\partial \vartheta}\right) - g\xi_\perp$$

:Space-derivative operator $\mathcal{L}$ has the same form as that in a static plasma

$$\bar{\rho} \equiv \frac{\mu_0 \rho |\mathbf{k}|^2 \sqrt{g}}{B^2}, \quad f \equiv \frac{|\mathbf{k}|^2}{B^2\sqrt{g}}$$

$$U \equiv \frac{2\mathbf{k}\cdot\nabla\Omega}{|\mathbf{k}|^2}$$

$$g \equiv -\frac{2\mu_0}{B^4}(\mathbf{B}\times\mathbf{k}\cdot\boldsymbol{\kappa})(\mathbf{B}\times\mathbf{k}\cdot\nabla p)$$

Two important features are:

(1) The coefficients of the equation depend on time through the wave vector,

$$\mathbf{k} \equiv \nabla\zeta - q\nabla\theta - (\vartheta - \tilde{\theta}_k(t))\nabla q \qquad \tilde{\theta}_k(t) \equiv \theta_k - \dot{\Omega}t \qquad \dot{\Omega} \equiv \mathrm{d}\Omega/\mathrm{d}q$$

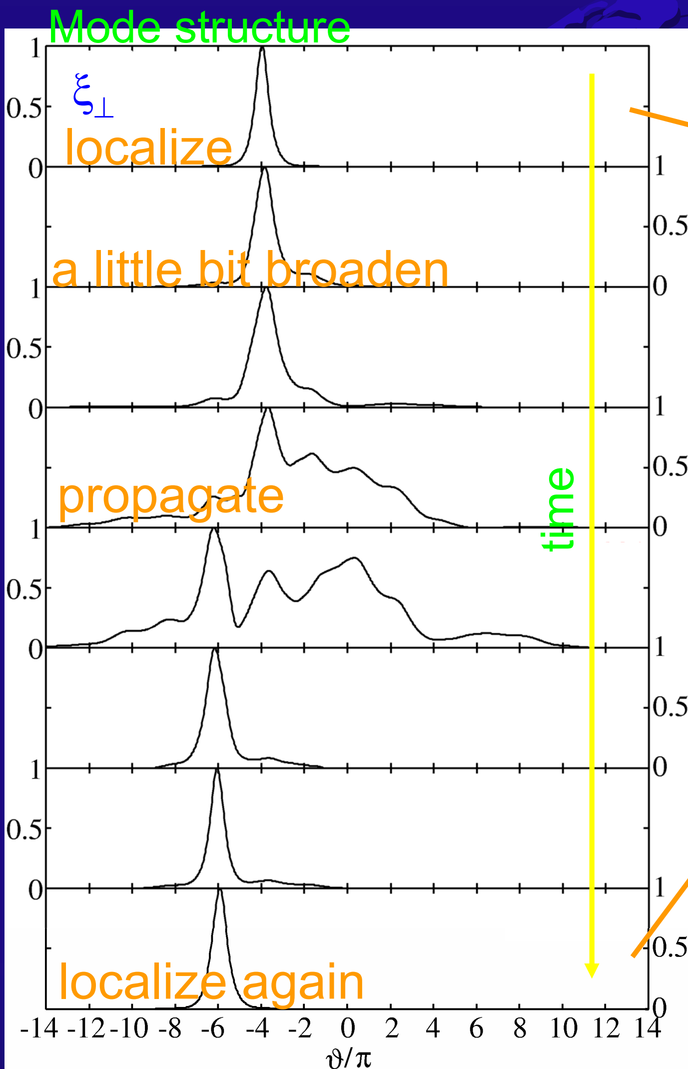
which is the same as the original ballooning equation

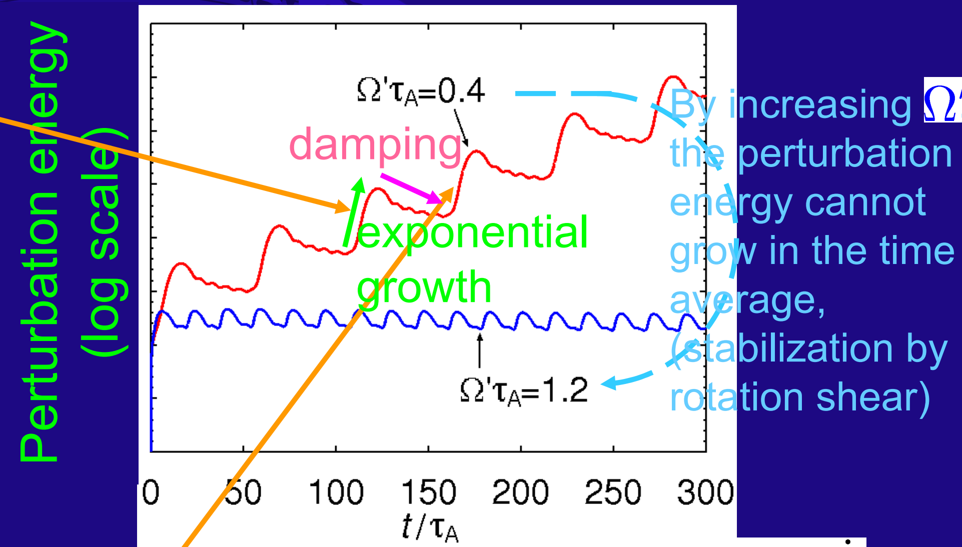
[F. L. Waelbroeck and L. Chen, Phys. Fluids B **3**, 601 (1991).]

(2) When the toroidal rotation shear is set to zero, the equation reduces to the incompressible ballooning equation

[J. W. Connor, R. J. Hastie, J. B. Taylor, Phys. Rev. Lett. **40**, 396 (1978).]

Time evolution of the perturbation





time evolution with period $\tau_d \equiv 2\pi/\dot{\Omega}$

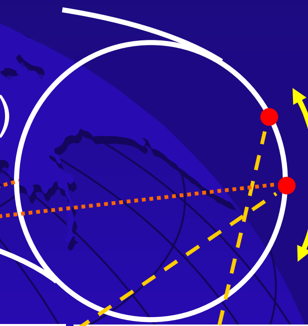
When Ω' is very small, the mode may evolve, in a short time scale, as an ideal MHD mode in a static plasma

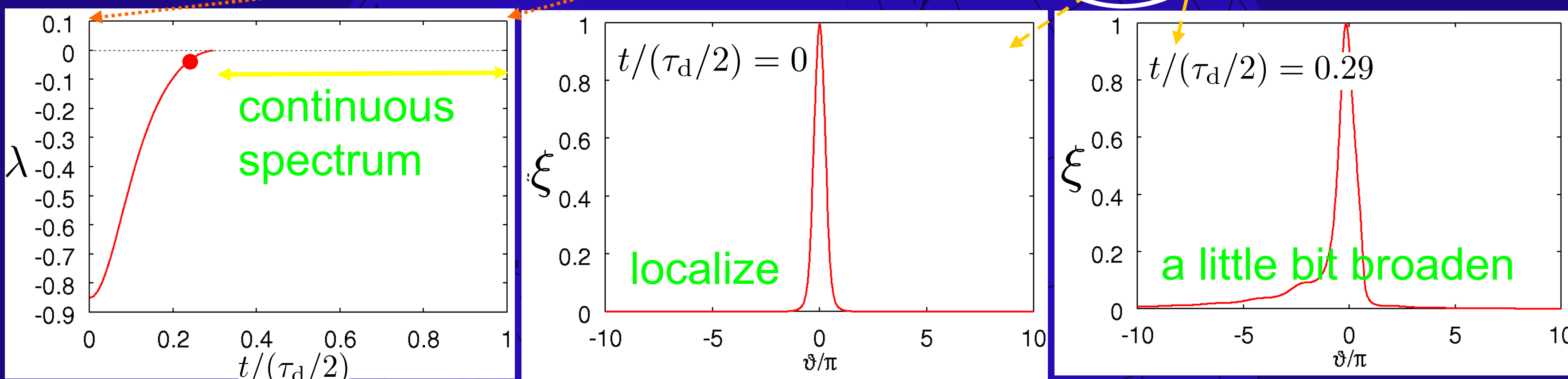
Therefore, the time evolution may have close relationship with the eigenvalue and the eigenfunction of the ballooning equation in a static plasma

Solution to the static ballooning equation

- **Static ballooning equation** $-\lambda\bar{\rho}\xi = \mathcal{L}\xi$ (λ : eigenvalue) **includes t as a parameter through $\mathbf{k}$**

$$\mathbf{k} \equiv \nabla\zeta - q\nabla\theta - (\vartheta - \tilde{\theta}_k(t))\nabla q \qquad \tilde{\theta}_k(t) \equiv \theta_k - \dot{\Omega}t$$





- **The localized eigenfunction and the growth rate are closely related to the mode structure and the instantaneous growth rate in the growing phase of the time evolution**
- **Therefore, it may be useful to expand the ballooning mode in a rotating plasma by the eigenfunctions of the static ballooning equation with the ballooning angle $\tilde{\theta}_k(t)$**

Difficulties in the eigenfunction expansion

- The time evolution of the ballooning mode in the rotating plasma and the eigenfunctions of the static ballooning equation seems to have close relationship

- Then, it may be useful to expand the ballooning mode in the rotating plasma by the eigenfunctions of the static ballooning equation for clarifying the stabilization mechanism

- However, the generalized eigenfunction belonging to the continous spectrum is not square-integrable ($\int_{-\infty}^{\infty} \mathrm{d}\vartheta \bar{\rho} |\xi|^2$ diverges)

- Therefore, we cannot treat it numerically

- We try to find a set of square-integrable eigenfunctions suitable for expanding the ballooning mode

Modification to the weight function

- **Spectrum of an eigenvalue problem is determined by the operator itself as well as the weight function and the boundary condition**

  e.g. Laplace operator :

  point spectrum for finite domain (Fourier series expansion)

  continuous spectrum for infinite domain (Fourier transform)

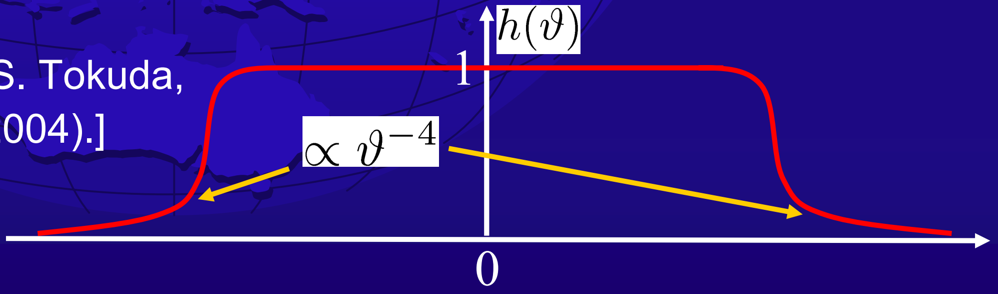
- **A modified eigenvalue problem:**

$$-\bar{\rho}h\lambda\xi = \mathcal{L}\xi \qquad \mathcal{L}\xi \equiv \frac{\partial}{\partial\vartheta}\left(f\frac{\partial\xi}{\partial\vartheta}\right) - g\xi$$

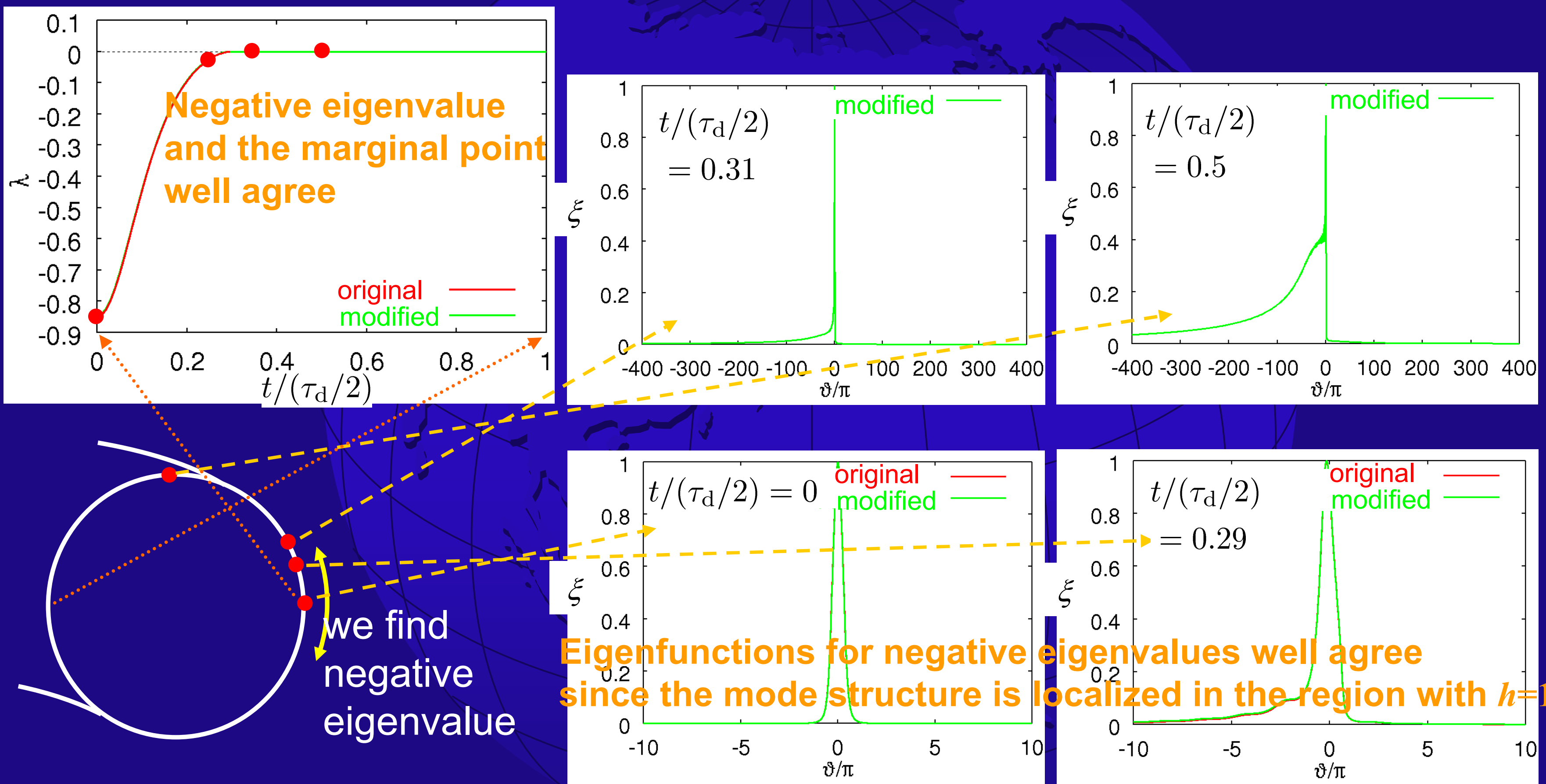
(ballooning equation: $-\bar{\rho}\lambda\xi = \mathcal{L}\xi$)

- **By the asymptotic analysis, we found that $h(\vartheta)$ has to be proportional to ϑ^{-4} at large $|\vartheta|$**

[M. Furukawa, Z. Yoshida and S. Tokuda, Submitted to Phys. Plasmas (2004).]



Resolution of the difficulty with the continuous spectrum



- We succeed to obtain the point spectrum and the *regularized* eigenfunctions at the stable side

Expansion of $\xi_\perp$ by the regularized eigenfunctions

- We expand the ballooning mode in the rotating plasma by the regularized eigenfunctions

$$\xi_\perp = \sum_j a_j(t)\xi_j(\vartheta, t) \qquad a_j = \int_{-\infty}^{\infty} \mathrm{d}\vartheta \bar{\rho} h \xi_j^* \xi_\perp$$

- Then we obtain coupled evolution equations for a_j's

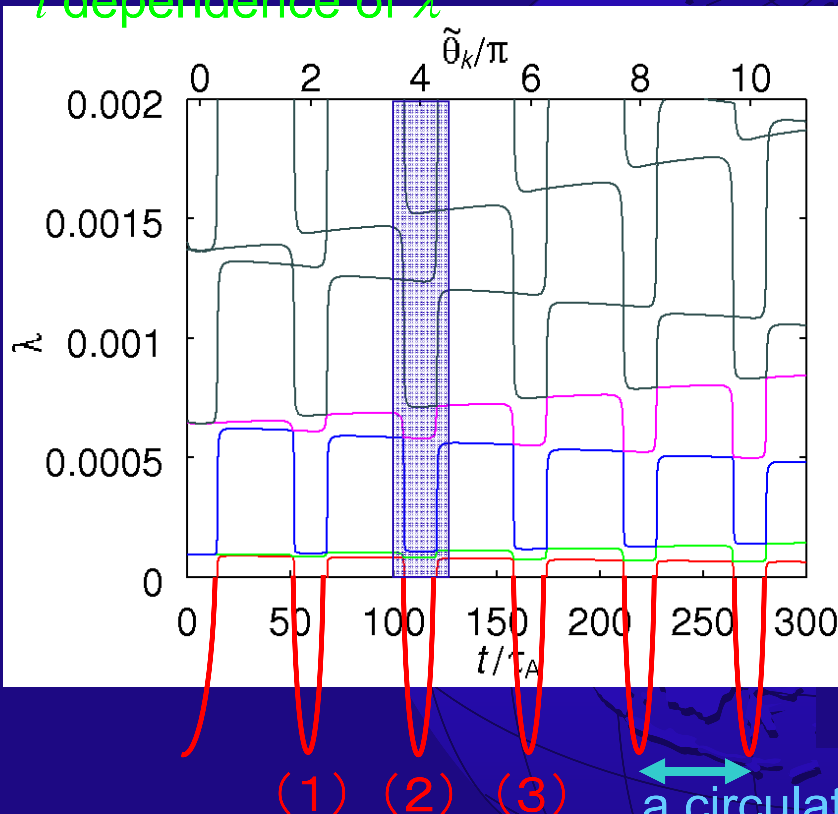
$$\frac{\mathrm{d}^2 a_j}{\mathrm{d}t^2} + \sum_k C_{1jk}\frac{\mathrm{d}a_k}{\mathrm{d}t} + \sum_k C_{2jk} a_k = -\sum_k \lambda_k C_{3jk} a_k$$

$$\begin{cases} C_{1jk} \equiv \int_{-\infty}^{\infty} \mathrm{d}\vartheta \bar{\rho} h \xi_j^* \left(2\frac{\partial \xi_k}{\partial t} - U\xi_k\right) \\ C_{2jk} \equiv \int_{-\infty}^{\infty} \mathrm{d}\vartheta \bar{\rho} h \xi_j^* \left(2\frac{\partial^2 \xi_k}{\partial t^2} - U\frac{\partial \xi_k}{\partial t}\right) \\ C_{3jk} \equiv \int_{-\infty}^{\infty} \mathrm{d}\vartheta \bar{\rho} h^2 \xi_j^* \xi_k \end{cases}$$

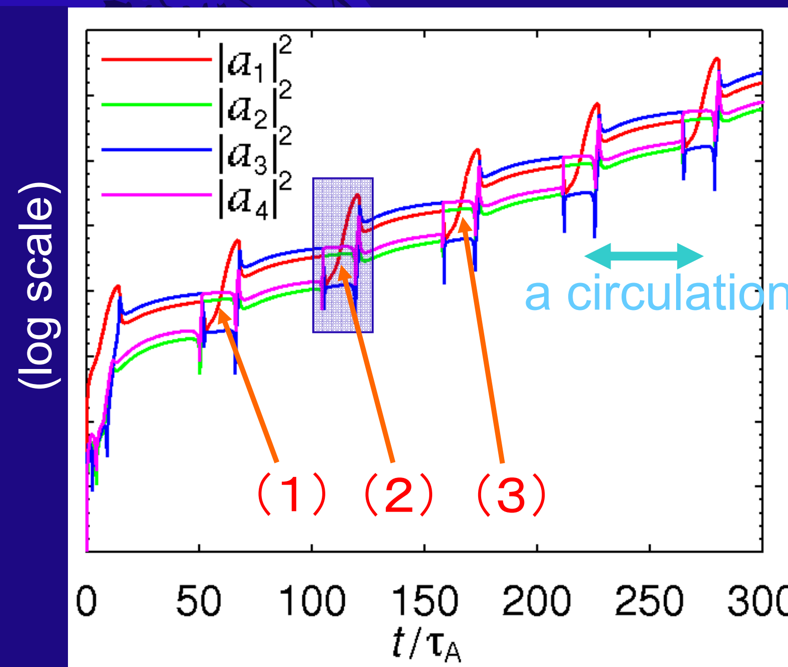
- The toroidal rotation shear makes couplings among a_j's through C_{1jk} and C_{2jk}

Time evolution of λ_j's and $|a_j|^2$'s

t dependence of λ



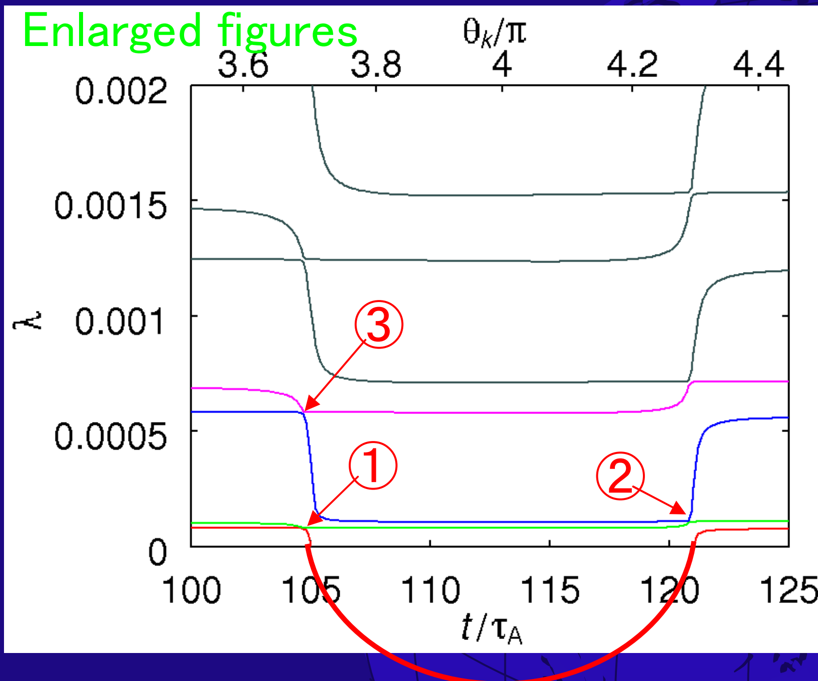
Time evolution of the expansion coefficient obtained from a numerical solution

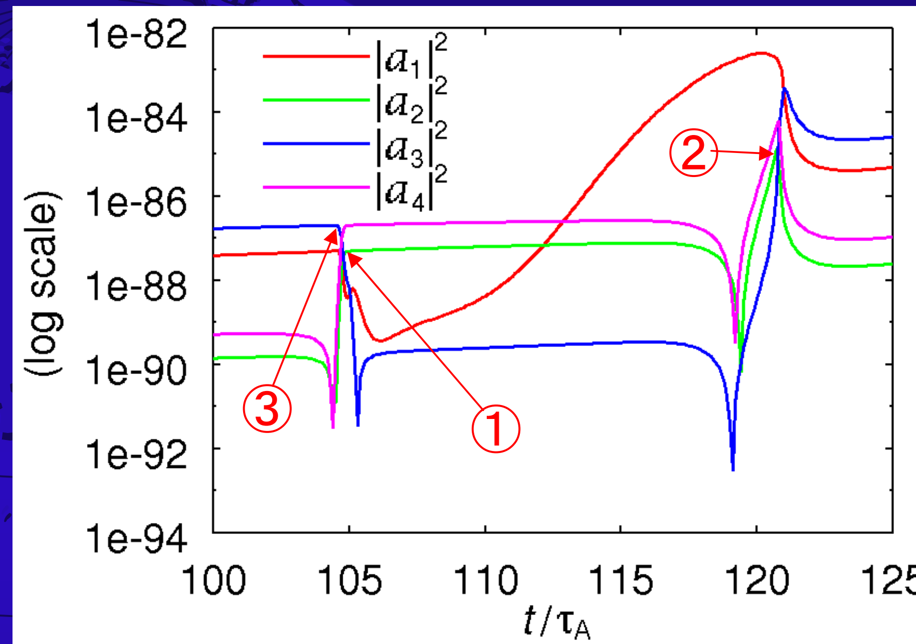


a circulation along a magnetic field line

- **Characteristic structure appears**
- **This originates from the torus geometry**

Energy transfer through mode couplings





- When the eigenvalues cross such as ①, ② and ③, the energy is transferred to higher (stable) modes successively
- The number of such crossings seems countably infinite during a finite duration
- Therefore, $|a_1|^2$ cannot grow in the time average, although it grows during $\lambda_1 < 0$

Background and Motivation (II)

- We have to solve an initial-value problem to identify whether a toroidally rotating tokamak equilibrium is stable or not
- If we have an analytic formula for estimating the ballooning stability in a toroidally rotating tokamak, it will significantly reduce computational costs
- We derive such a formula on the basis of the physics of stabilization of ballooning modes

Time trace of the mode evolution

exponentially growing | damping

t: 0 — $\Theta/\dot{\Omega}$ — $\pi/\dot{\Omega}$

θ_k: 0 — Θ — π

time evolution

Assume the rotation shear is small. Couplings among the eigenmodes are weak. Then, perturbation energy grows as same as the unstable eigenmode

$$\propto \exp\left[2\int \mathrm{d}t\gamma^{\mathrm{st}}(t)\right]$$

perturbation energy seems to damp by phase mixing

$$\propto \frac{1}{t^2}$$

perturbation energy

$$\tilde{E}(0)$$

$$\tilde{E}(0)\exp\left[\frac{2}{|\dot{\Omega}|}\int_0^{\Theta}\mathrm{d}\theta_k\gamma^{\mathrm{st}}(\theta_k)\right]$$

$$\tilde{E}(0)\exp\left[\frac{2}{|\dot{\Omega}|}\int_0^{\Theta}\mathrm{d}\theta_k\gamma^{\mathrm{st}}(\theta_k)\right]\left(\frac{\Theta}{\pi}\right)^2$$

Then we obtain stability criterion as:

$$|\Omega'| \geq \frac{|q'|\int_0^{\Theta}\mathrm{d}\theta_k\gamma^{\mathrm{st}}(\theta_k)}{\ln(\pi/\Theta)}$$

$\gamma^{\mathrm{st}}(\theta_k)$ and Θ are obtained by solving just an ODE

Semi-analytic formula

- **Assume** $\gamma^{\rm st}(\theta_k) = \gamma_{\max}\left(1 - \frac{\theta_k^2}{\Theta^2}\right)$
  then we obtain

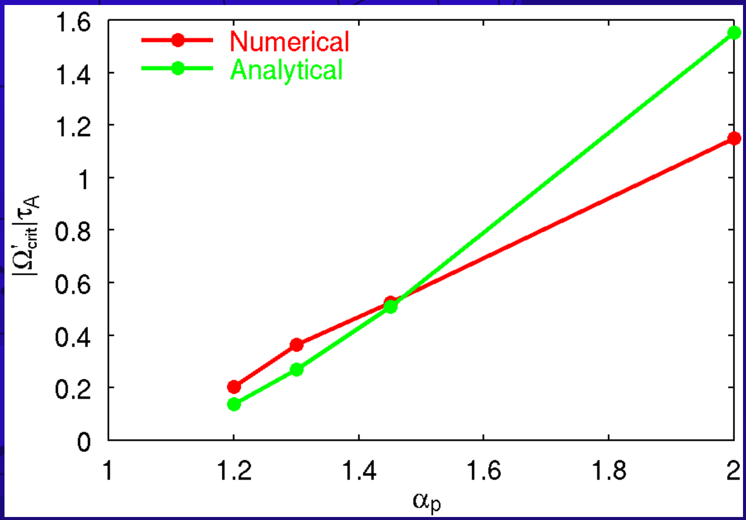
$$|\Omega'| \geq \frac{2|q'|\gamma_{\max}\Theta}{3\ln(\pi/\Theta)}$$ for stability

γ_{max} and Θ are obtained by solving an ODE twice

- **Small toroidal rotation shear can stabilize the ballooning mode when γ_{max} and Θ are small, which can be controlled by geometrical effects such as D-shaping**

- **If q' is varied, γ_{max} and Θ also change**

- **The analytic formula roughly agrees with numerical results**



Conclusions

- **Toroidal rotation shear generates countably infinite number of crossings among eigenvalues during a finite duration**
- **When the crossings occur, the energy is transferred to higher (stable) modes**
- **Therefore, an unstable mode cannot grow in the time average**
- **This energy transfer works as the stabilization mechanism of ballooning modes**
- **Simple semi-analytic formula is derived for estimating toroidal rotation shear required to stabilize the ballooning mode, which will significantly reduce computational costs**