THE GATO CODE: ALL YOU EVER WANTED TO KNOW AND A HELLUVA LOT MORE

A.D. Turnbull, General Atomics

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GATO Is A Linearized Ideal MHD Stability Code Based On Variational Energy Principle of Bernstein, et al., 1958

- Minimize Lagrangian $L = W - \omega^2 K$ over all permissible (physically meaningful) trial displacements ξ: $\Rightarrow \delta W(\xi^+,\xi) - \omega^2 \delta K(\xi^+,\xi) = 0$

- This can be considered as an eigenvalue problem for eigenvalue $\lambda = \omega^2$ and eigenfunction ξ

- For any positive definite $\delta K(\xi^\dagger, \xi)$ then:

$$\lambda = \omega^2 = \delta W / \delta K$$

  is the Rayleigh Quotient and its sign determines the stability

$$\begin{cases} \lambda = \omega^2 < 0 \Rightarrow \text{unstable } (\omega = i\gamma) \\ \lambda = \omega^2 > 0 \Rightarrow \text{stable} \end{cases}$$

- Use the projection:

$$X = \xi \cdot \nabla\psi \qquad Y = r/B_\phi \, \xi \cdot \nabla\phi \qquad J = (\nabla\psi \cdot \nabla\chi \times \nabla\phi)^{-1}$$

$$U = \xi \cdot (\nabla\psi \times \nabla\phi) / B_p - Y + J\beta_\chi X \qquad \left(\beta_\chi = \nabla\psi \cdot \nabla\chi / |\nabla\psi|^2 \right)$$

in a nonorthogonal (ψ,χ,ϕ) coordinate system

Ritz-Galerkin Method Converts Minimization To A Matrix Eigenvalue Problem For Coefficient Amplitudes

- **GATO solves for eigenmodes using Finite Hybrid Element Galerkin expansion of X, U, and Y in both radial and poloidal directions:**
  - Finite Hybrid Elements (FHE) in the radial ψ and poloidal χ directions and
  - Fourier series in the toroidal ϕ direction:
    - Uncoupled in case of underlying axisymmetric equilibrium

$$A\,X = \lambda\,B\,X$$

$$X \equiv \begin{pmatrix} X_{0,0} \\ \vdots \\ U_{j,i} \\ Y_{j,i} \\ X_{j,i} \\ U_{j,i+1} \\ Y_{j,i+1} \\ X_{j,i+1} \\ \vdots \\ U_{j,M} \\ Y_{j,M} \\ X_{j,M} \end{pmatrix} \qquad j = 1, N$$

  - Vector ***X*** contains finite element node values of X, U, and Y
  - Matrices ***A*** and ***B*** represent the potential and kinetic energy or other appropriate norm
  - System is solved for ***X*** from which the eigenvector ξ can be reconstructed

The Finite Hybrid Element Method Is A Modification Of The Standard Finite Element Method Responses

- **Finite Elements:**
  - Expand X, U, and Y in Finite Elements $\delta W = \delta W(X, U, Y)$
  - Numerical differentiation for terms: $\partial X/\partial\psi, \partial X/\partial\chi, \partial U/\partial\chi, \partial Y/\partial\chi$
  - Minimize δW with respect to coefficients of: $X, U, \text{and } Y$

- **Finite Hybrid Elements:** $\delta W = \delta W(X, \partial X/\partial\psi, \partial X/\partial\chi, U, \partial U/\partial\chi, \partial Y/\partial\chi, Y)$
  - Expand $X, \partial X/\partial\psi, \partial X/\partial\chi, U, \partial U/\partial\chi, \partial Y/\partial\chi, Y$ in Finite Elements
  - Impose additional constraints linking: $\begin{cases} X \text{ with } \partial X/\partial\psi \text{ and } \partial X/\partial\chi \\ U \text{ with } \partial U/\partial\chi \text{ and } Y \text{ with } \partial Y/\partial\chi \end{cases}$
  - Minimize δW with respect to coefficients of $X, \partial X/\partial\psi, \partial X/\partial\chi, U, \partial U/\partial\chi, \partial Y/\partial\chi, Y$

Hybrid Elements Avoid Numerical Pollution of The Spectrum

- **Precise evaluation of marginal stability requires:**

$$\nabla \cdot \xi \sim \frac{\partial \xi_\psi}{\partial \psi} + m\xi_\chi + n\xi_\phi \to 0$$

- **If *X*, *U*, and *Y* are linear tent elements then ∂*X*/∂ψ, ∂*X*/∂χ, etc. will be constant elements and ∇•ξ cannot vanish everywhere**

- **This can only happen everywhere**
  - Not just at nodal points -

  if each term has the same functional form:

$$\nabla \cdot \xi \sim \frac{\partial \xi_\psi}{\partial \psi}, \xi_\chi, \quad \text{and } \xi_\phi \text{ are the same kind of element}$$

Hybrid elements allow all components of ∇•ξ to be of same type

- But at some cost: stable continuum is numerically destabilized

GATO Is Designed To Treat Ideal MHD Stability Of Axisymmetric Equilibria In Full Toroidal Geometry

- **Non-circular strongly shaped cross section includes:**
  - Up-down asymmetry
  - Divertor or Limiter plasma edge
  - Arbitrary aspect ratio including Spheromaks and RFPs
  - Arbitrary shaped surrounding wall
- **Toroidal mode numbers n = 0, 1, 2, 3, 4:**
  - Extending to intermediate n ~ 10 depending on circumstances
- **Arbitrary profiles:**
  - Negative shear, nonmonotonic q
  - Negative q
  - Strong gradients (H-mode and ITBs)
- **Some restrictions apply:**
  - Toroidal mode number limited by resolution but restricted to $n < 15$ due to breakdown in vacuum calculation
  - Wall required to be toroidally and poloidally continuous with no sharp corners

GATO Consists Of Four Separate Sources That Run Sequentially And Link Through Binary Data Files

SMAP: Mapping from equilibrium to optimally packed flux surface grid

SVAC: Construct matrices A (Potential energy matrix) and B (Kinetic energy matrix)
Construct wall and vacuum

SEIG: Solve eigenvalue equation for eigenvalue λ and eigenmode *X*

$$A X = \lambda B X$$

SPLT: Reconstruct physical eigenmode from Finite element node values in *X*
Plotting and diagnostics

- Sources are generally labeled as: sxxxhyyy.f with $xxx = \begin{Bmatrix} map \\ vac \\ eig \\ plt \end{Bmatrix}$ and $yyy = \begin{Bmatrix} 100 \\ 200 \\ 300 \\ 400 \end{Bmatrix}$

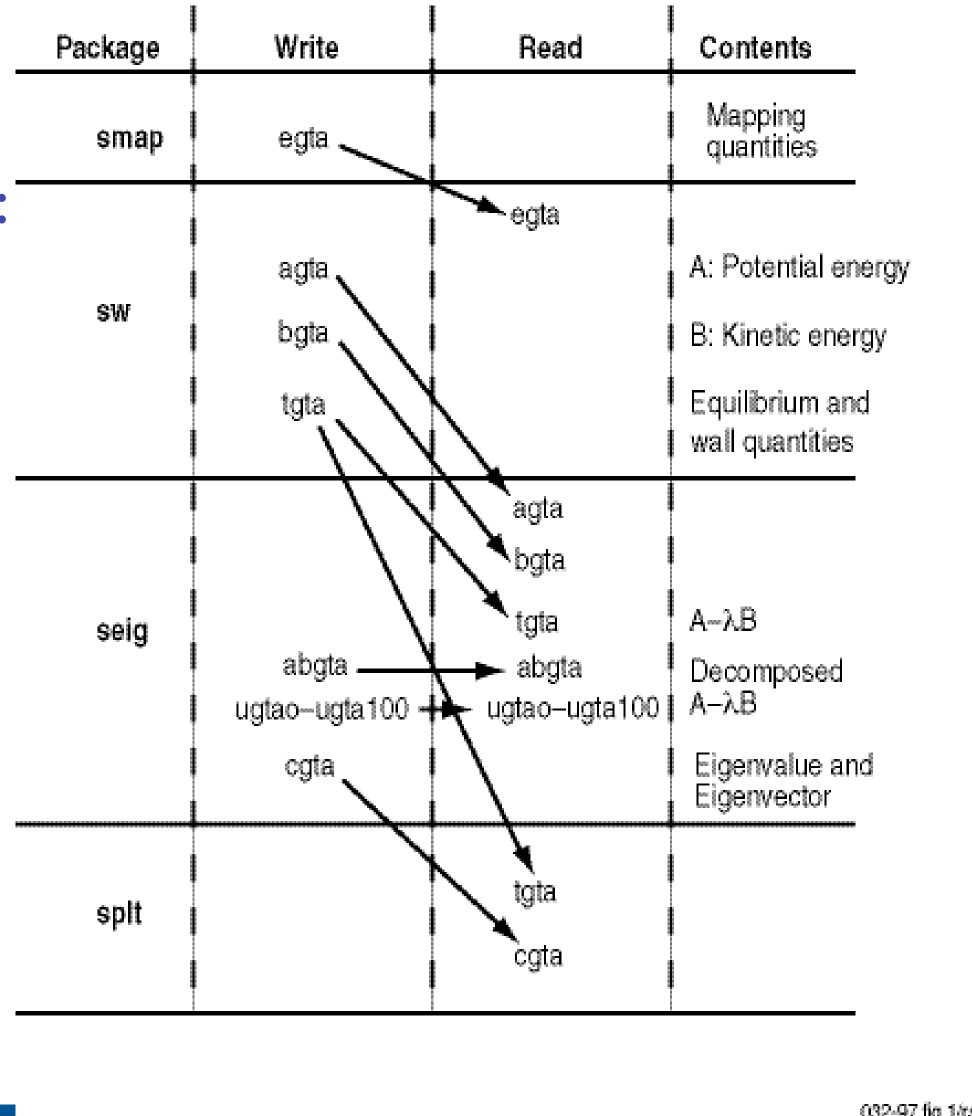
where: $N_\psi = yyy \quad N_\chi = 2yyy$ are the precompiled mesh sizes

Input And Output Is Provided Mostly Through ASCII Files

- **Input is from three files:**
  - ‘*eqgta*’: Equilibrium (EFIT ‘g file’, TOQ ‘dskgato file’ or JSOLVER ‘u-file’
  - ‘*ingta*’: Namelist input
  - ‘*inwgta*’: Namelist input for wall data
- **Namelist input ‘*ingta*’ is read by all four sources:**
  - Equilibrium ‘*eqgta*’ file is read by mapping SMAP only
  - Namelist input ‘*inwgta*’ is read by SVAC only and only if a finite wall
- **Output is to ASCII and to cgm or PostScript files:**
  - ASCII files labeled ‘*okgta*‘ for *k* = *1, 2, 3, 4* from the four sources
  - Graphics output from SMAP and SPLT labeled ‘*gato1.cgm*’ and ‘*gato4.cgm*’
  - Graphics metafiles (cgm) can be converted to PostScript
- **Additional equilibrium and eigenmode output produced by SPLT:**
  - ‘*vacuum*’: Input for Vacuum analysis (ASCII)
  - ‘*diagnostics*’ Limited input for graphics analysis (ASCII)
  - ‘*nimrod*’: Limited input for code benchmarks (Binary)
  - ‘*o4dump*’: Complete input for graphics analysis (ASCII)

Link Between Sources Is Through Binary Data Files

- **Sources are linked by a set of binary files labeled ‘*xgta*’ where:**
  - x = *a,b,e,c,t,u*
- **Sources can be run various ways:**
  - Manually:
    - Requires manual deletion of large binary files at end of run
  - Through a set of standard scripts:
    - *runGATO_label.batch (controls the specific run)*
    - *runGATO_version.script (called by runGATO_label.batch and does all file handling*
  - Through a set of alternative scripts available in /d/gato/
  - From the FusionGrid
- **Scripts typically rename the ASCII and cgm output files:**
  - Generally as ‘*outpt.xxxx*’



Several Unique Features Allow High Accuracy And Flexibility In Applications

- **Mapping from either direct or inverse equilibria**
- **Choice of Jacobian:**
  - Equal-arc or Straight Field line PEST (restricted wall options)
- **Extremely flexible and optimized grid packing:**
  - Automatic well-optimized packing at rational surfaces
  - Additional packing at edge or magnetic axis
  - Additional packing at prescribed q values or ψ values
  - Additional options for nonmonotonic q
- **Flexible conducting wall options:**
  - Wall on the plasma and no wall options
  - Real wall input as:
    - Set of (r_{wall}, z_{wall}) coordinates or
    - Set of fourier coefficients in poloidal angle for r_{wall} and z_{wall}
  - Built in wall constructions:
    - Conformal wall (constant normal distance)
    - Self-similar wall (constant expansion factor)
  - Expansion factor available for all wall options
- **Low memory requirements:**
  - Trade off between memory and disk space

Features Allow Additional Physics Within Ideal MHD Model

- **Choice of Kinetic Energy norm:**
  - Full kinetic energy or incompressible
  - Physical (measured) or model density profiles
  - Full eigenvector or δW code (sign of eigenvalue)
- **Mode selection for most unstable mode or any other mode**
- **Array of final mode diagnostics:**
  - Normal orthogonal representation: $\xi_\psi = \xi \cdot \nabla\psi / |\nabla\psi|^2$ $\quad \xi_p = r\xi \cdot (\nabla\psi \times \nabla\phi) / |\nabla\psi|$ $\quad \xi_t = r\xi \cdot \nabla\phi$
  - The GATO representation: $X = \xi \cdot \nabla\psi$ $\quad U = \xi_p / B_p - Y + J\beta_\chi X$ $\quad Y = r / B_\phi \, \xi \cdot \nabla\phi$
  - Cylindrical representation: $\xi_r = \xi \cdot \nabla r$ $\quad \xi_z = \xi \cdot \nabla z$ $\quad \xi_\phi = r\xi \cdot \nabla\phi$
  - Field line representation: $\xi_n = \xi \cdot \nabla\psi / |\nabla\psi|^2$ $\quad \xi_\perp = \xi \cdot (\nabla\psi \times B) / (|B||\nabla\psi|)$ $\quad \xi_B = \xi \cdot B / |B|$
- **Similar decompositions for δB and δA = ξ ×B**
- **Array of plot types:**
  - Displacement vectors, line plots, Fourier analysis, contour plots
- **Array of radial and poloidal coordinates for analysis:**
  - Independent of coordinates used in actual calculation

Recent Upgrades Have Added Important Features And Made Code Much More Robust

- **Mapping of plasma boundary:**
  - Account taken of X-point in mapping of boundary
  - Error handling and correction with automatic retry
  - Search for last closed flux surface implemented
- **Additional packing flexibility:**
  - Independent packing for negative or positive shear region only
- **Greatly enhanced diagnostics:**
  - All representations of ξ δB and δA for all plot types
  - Flexibility for switching off or on
  - Independent switching of individual components
  - All major radial and poloidal coordinates:

$$\rho_{midplane}\ ,\psi_{pol}\ ,\Psi_{tor}\ ,V \qquad \rho^2_{midplane}\ ,\sqrt{\psi_{pol}}\ ,\sqrt{\Psi_{tor}}\ ,\sqrt{V}$$

Normalized and un-normalized

- **Robust eigenvalue search procedure:**
  - Simple check for sign of δW
  - Limits on search to prevent runaway iterations
  - Partial restart options
- **Additional δW norm using only normal component of ξ**

$$\delta K = \int \rho|\xi_\psi|^2 d\tau \quad \text{or} \quad \delta K = \int \rho|X|^2 d\tau$$

Most Important Recent Upgrade Allows Continuum To Be Shifted Back To Stable Side Of Marginal Point

- **Previously the edge of the continuum was numerically destabilized by the additional flexibility of the Hybrid Elements:**
  - For a finite mesh they converged to marginal stability from unstable side
  - Required some interpretation of result in case of standard (full) kinetic energy norm
  - Led to serious numerical pollution of spectrum for alternative norm
- **Analysis of numerical implementation of localized modes yields estimate of numerical destabilization:**
  - Following Degtyarev, Medvedev et al. (1986), numerical destabilization term is estimated by analysis of the form of δW for displacements radially localized around a rational surface ψ_0 where m - n q(ψ_0) = 0:

$$\xi \sim x^{-1/2-\mu} = \left|\psi - \psi_0\right|^{-1/2-\mu}$$

$$\text{Then} \quad \delta W \sim \int \left[\frac{1}{\left\langle |B|^2 / |\nabla\psi|^2 \right\rangle} \left| x\langle S\rangle \frac{\partial \xi}{\partial x} - \langle T\rangle \xi \right|^2 - R|\xi|^2 \right] dx$$

Greene, Johnson 1962

$$S = \left(\frac{B\times\nabla\psi}{|\nabla\psi|^2}\right)\bullet\left[\nabla\times\left(\frac{B\times\nabla\psi}{|\nabla\psi|^2}\right)\right] \quad \text{is the local shear} \qquad T = \frac{j\cdot B}{|\nabla\psi|^2} - S$$

$$(J \equiv \sqrt{g} \equiv \left|\nabla\psi\cdot\nabla\chi\times\nabla\phi\right|^{-1})$$

$$\langle Q\rangle = \frac{1}{4\pi^2}\oiint QJ\,d\chi\,d\phi \quad \text{is a flux surface average}$$

Application to Ideal MHD δW Yields A Numerical Correction To Be Subtracted Dependent on Local Shear

- **Applying correction to Greene and Johnson form:**
  - $\Rightarrow$ Numerical correction term is:

$$\delta W_c = -\frac{1}{2}\int\left[\frac{\frac{1}{8}h_\psi^2}{\left\langle |B|^2/|\nabla\psi|^2\right\rangle}\left[2\langle S\rangle\langle T\rangle + \langle S\rangle^2\right]\left|\frac{\partial\xi_\psi}{\partial\psi}\right|^2 d\psi\right] = -\frac{1}{2}\int\left[\frac{\frac{1}{4}h_\psi^2}{\left\langle |B|^2/|\nabla\psi|^2\right\rangle}\langle S\rangle\langle T + \tfrac{1}{2}S\rangle\left|\frac{\partial\xi_\psi}{\partial\psi}\right|^2 d\psi\right]$$

- **Local Shear can be evaluated in terms of equilibrium quantities:**

$$S = \frac{f'}{r^2} - \frac{f}{r^2|\nabla\psi|^2}\left(\frac{\partial}{\partial\psi}(|\nabla\psi|^2)_v - \Delta^*\psi\right) \qquad T = \frac{f}{r^2|\nabla\psi|^2}\left(\frac{\partial}{\partial\psi}(|\nabla\psi|^2)_v - 2\Delta^*\psi\right)$$

- **Flux surface integrals are then:**

$$\langle S\rangle = f'\mathfrak{A} - f\mathfrak{B} - f'p'\mathfrak{C} - f^2 f\mathfrak{D}$$

$$\langle T + \tfrac{1}{2}S\rangle = \tfrac{1}{2}f\mathfrak{A} + \tfrac{1}{2}f\mathfrak{B} + \tfrac{3}{2}f'p'\mathfrak{C} + \tfrac{3}{2}f^2 f'\mathfrak{D}$$

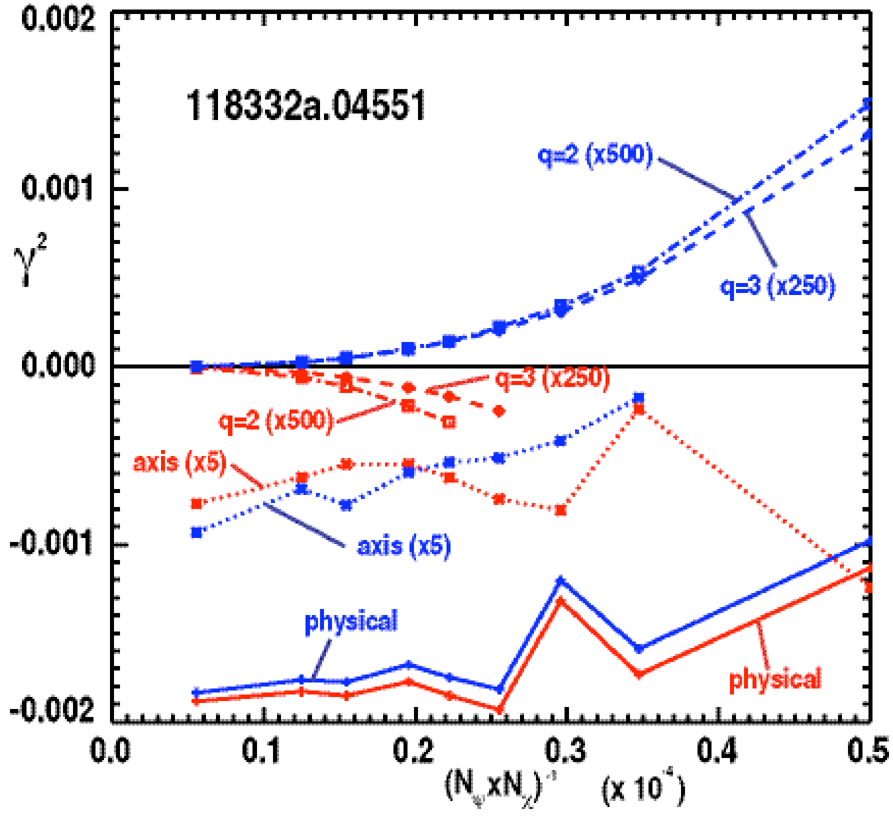
$$\mathfrak{A} = \left\langle \frac{1}{r^2}\right\rangle \quad \mathfrak{B} = \left\langle \frac{\frac{\partial}{\partial\psi}(|\nabla\psi|^2)_v}{r^2|\nabla\psi|^2}\right\rangle$$

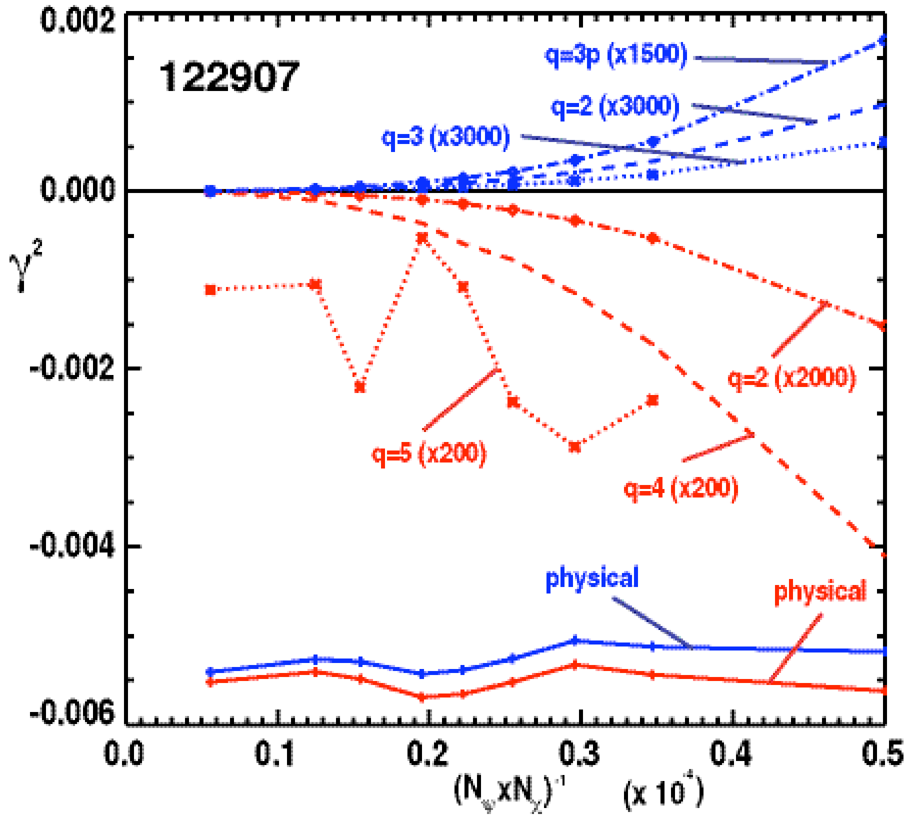
$$\mathfrak{C} = \left\langle \frac{1}{r^2|\nabla\psi|^2}\right\rangle \quad \mathfrak{D} = \left\langle \frac{1}{r^2|\nabla\psi|^2}\right\rangle$$

- **Note that:** $\langle S\rangle = \dfrac{dq}{d\psi}$

Convergence Of Physical Modes Unchanged: Continua Now Converge To Marginal Stability From Stable Side

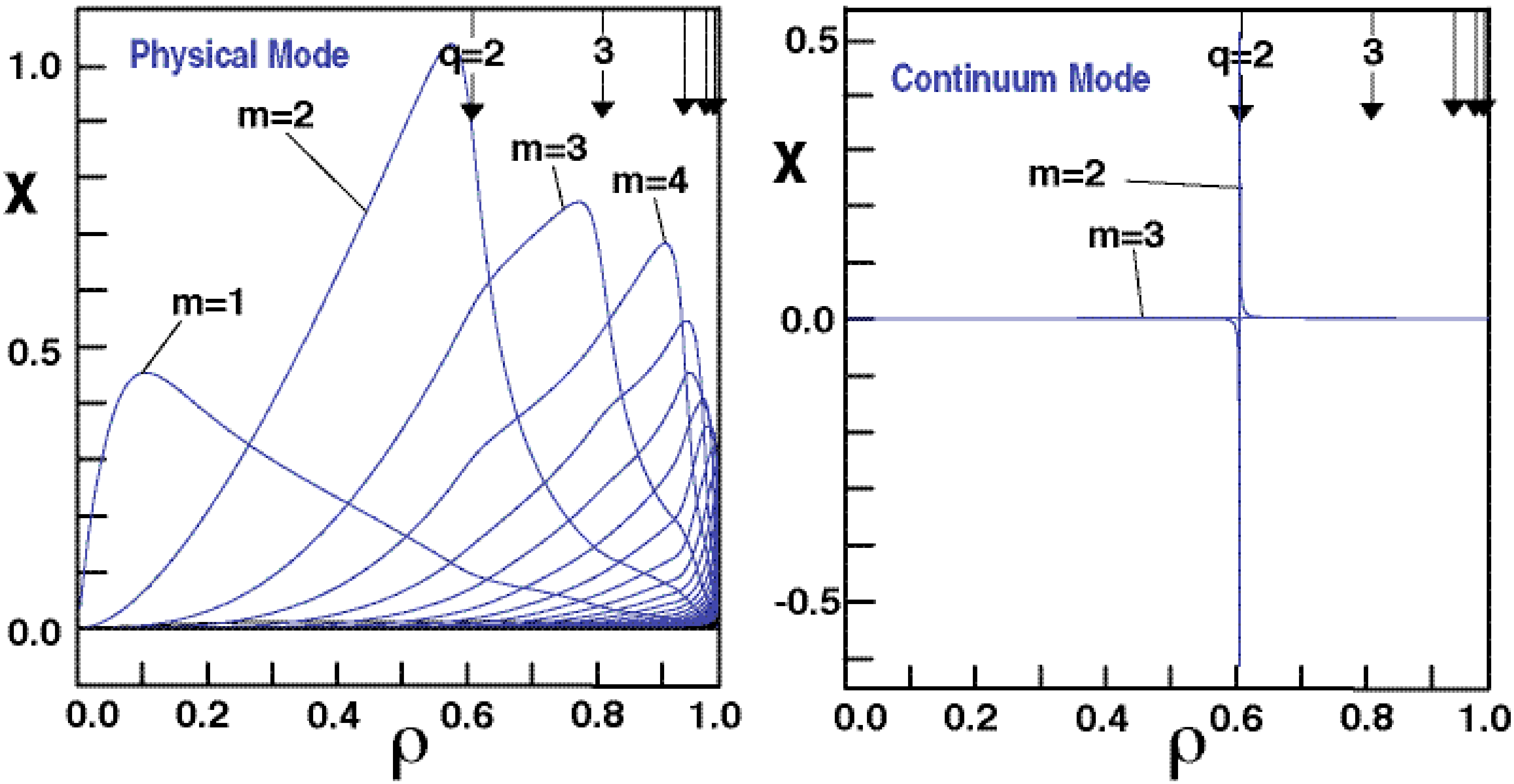
- **All modes converge to same point as without correction:**
  - Convergence remains quadratic





Numerical Correction Successfully Restabilizes Continuum Modes In All Cases So Far Considered

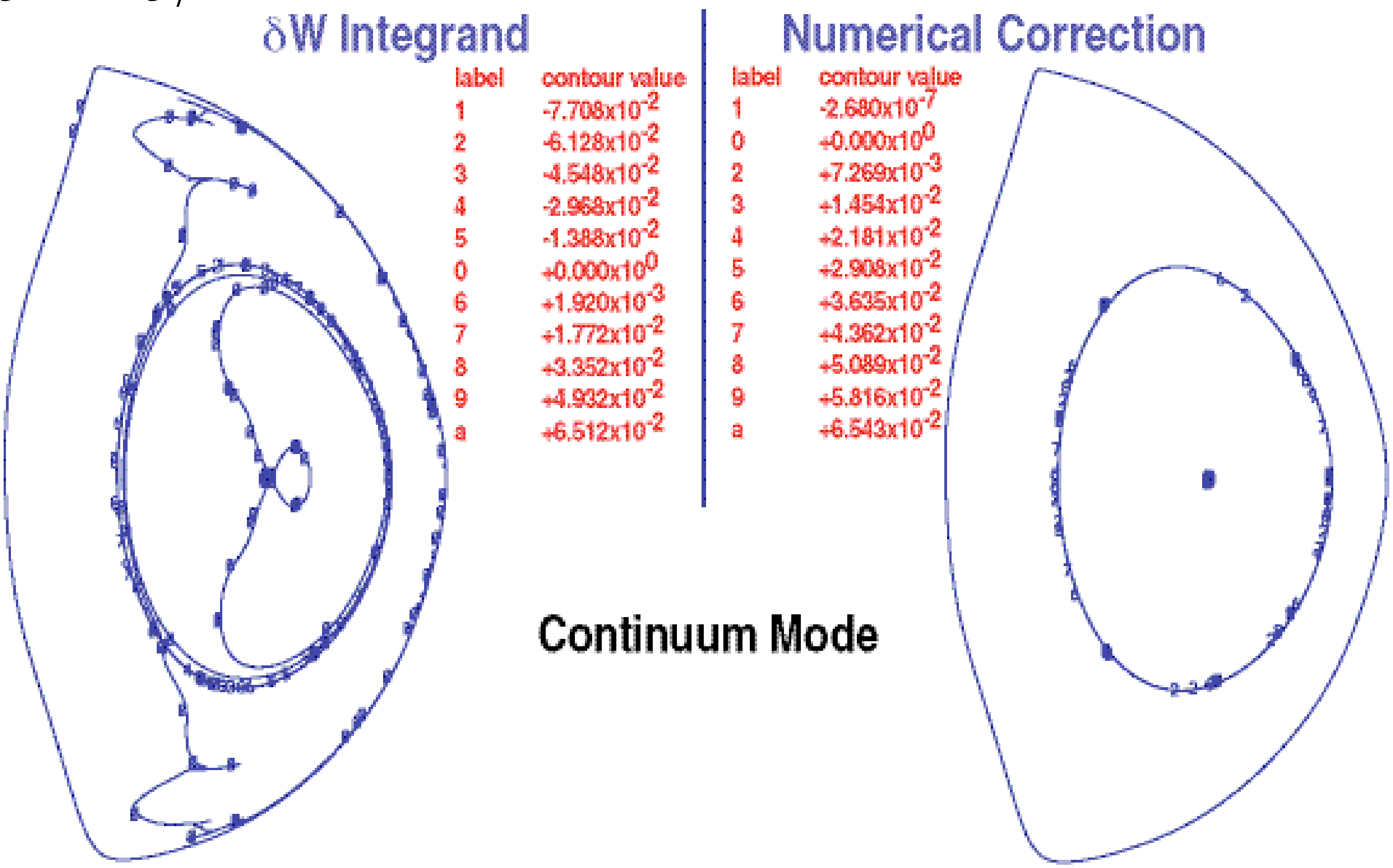
- Equilibrium with single physically unstable mode plus stable continua



Correction Convolved With Localized Displacements Provides Correction of Order of Numerical Eigenvalue

- δW_c can be significant even for small correction factor C

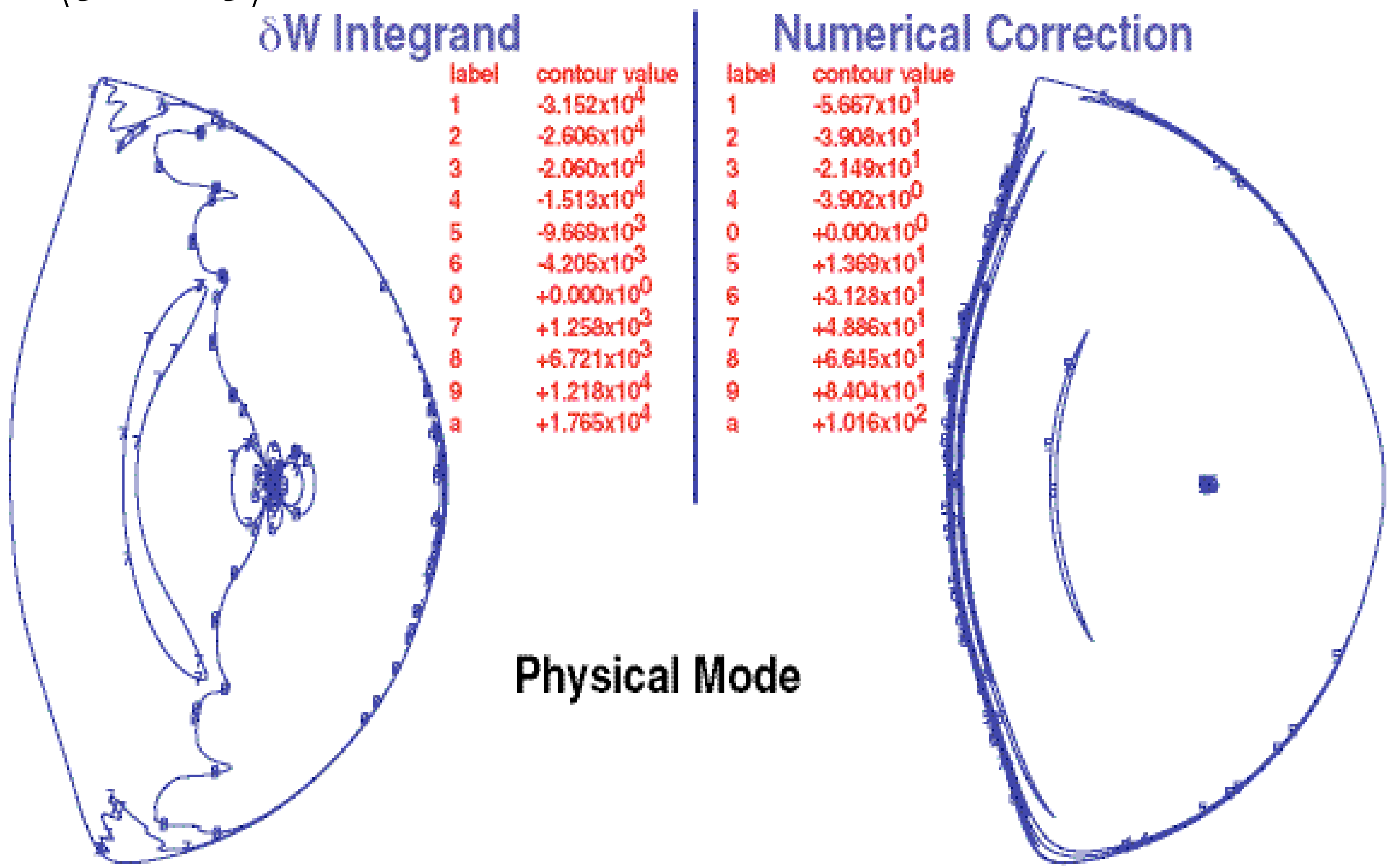
$$\delta W_c \sim \left\langle \xi^+ \cdot C \cdot \xi \right\rangle$$



Correction Convolved With Localized Displacements Is Negligible For Physical Displacements

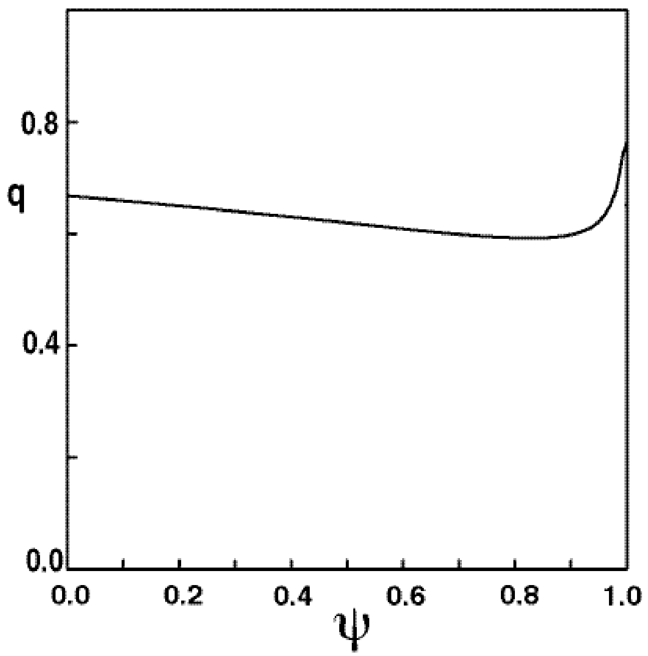
- **For same correction factor C δW_c is much smaller than the total δW**

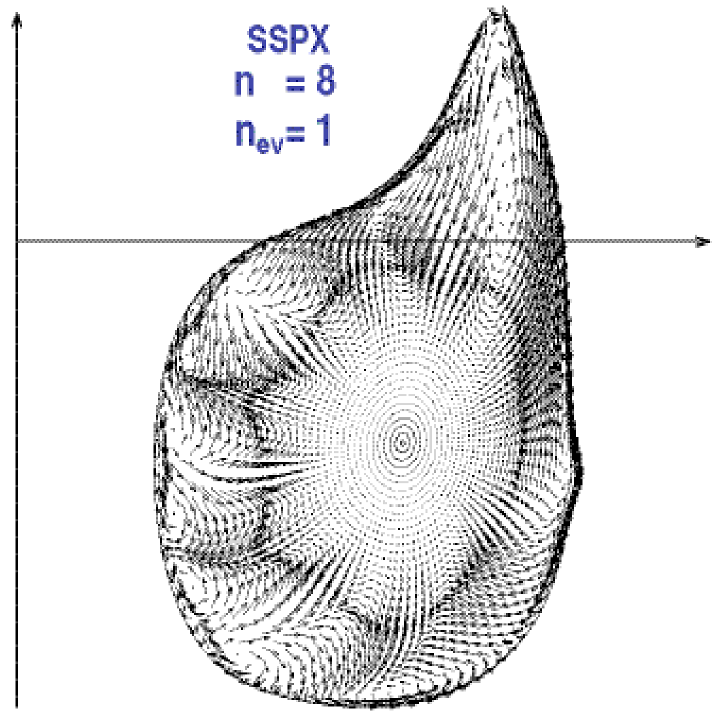
$$\delta W_c \sim \langle \xi^+ \cdot C \cdot \xi \rangle$$



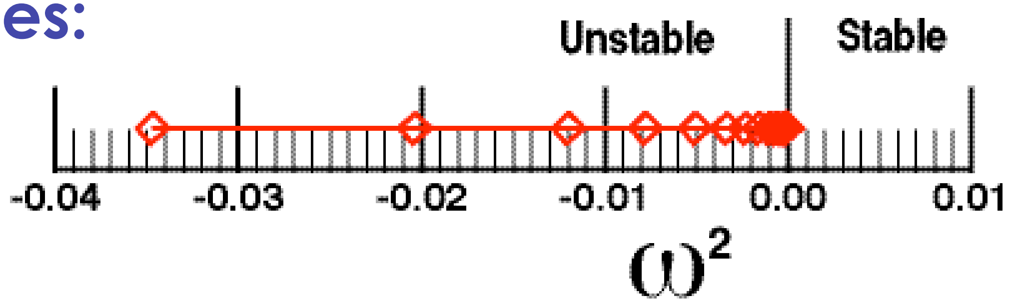
Application: Spheromak — Spectrum For SSPX Spheromak Shows A Sequence of Unstable Eigenmodes

- n = 8 pressure driven mode in SSPX



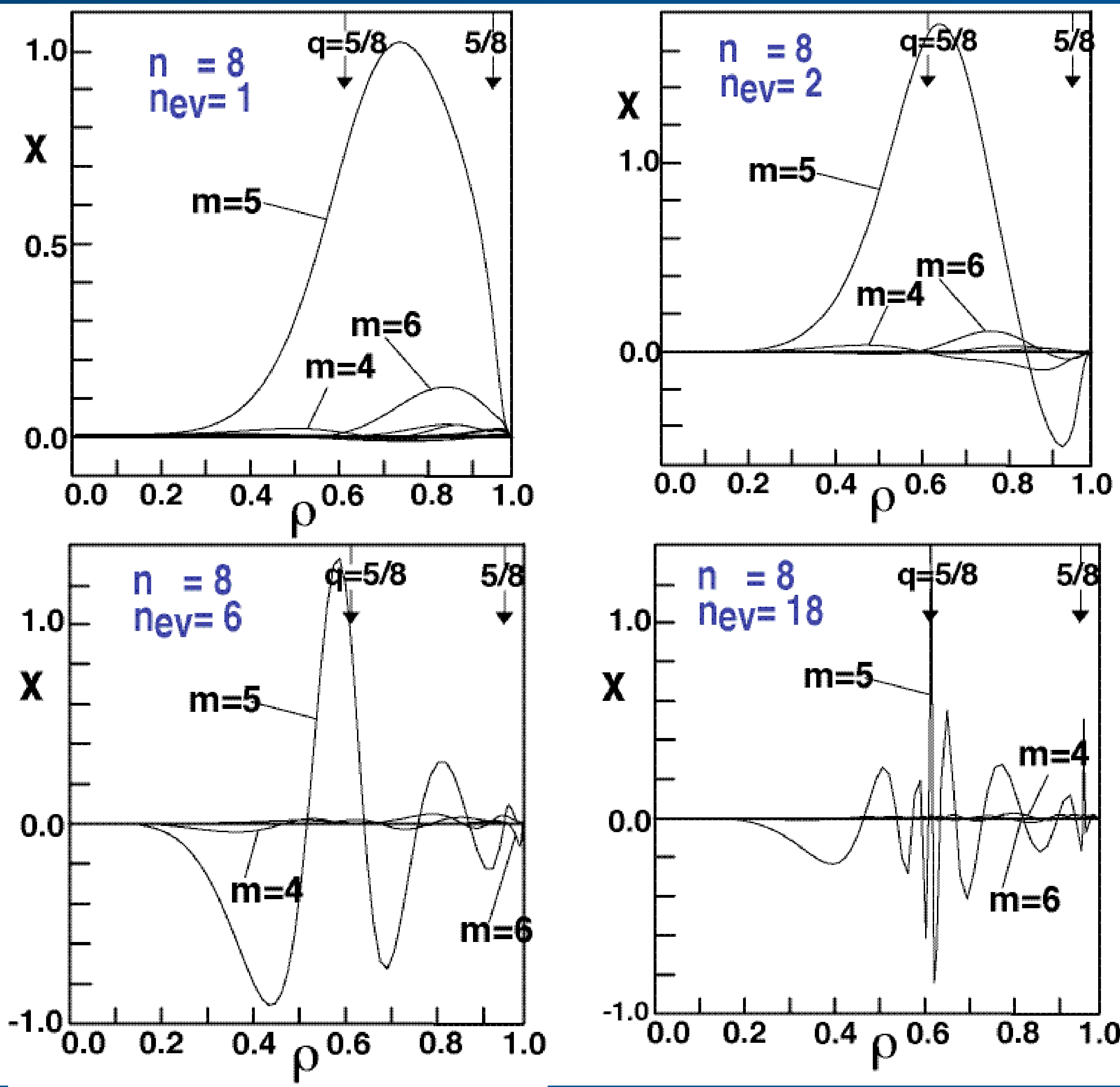


- Spectrum of unstable modes:
  - Apparent accumulation point at marginal stability



Application Spheromak: Unstable SSPX Spectrum For n = 8 Shows Sturmian Behavior

- **Restabilized continuum allowed multiple unstable modes to be more easily seen**
- **Spectrum exhibits Sturmian behavior:**
  - Successive modes increase number of oscillations
  - Oscillations period becomes short near resonant surface



GATO Is Well Tested And Has Been Shown To Provide Accurate Results In A Wide Variety Of Real Cases

- **GATO is extremely flexible and can treat the important features of real equilibria, including:**
  - Different toroidal devices:
    - Conventional and Advanced Tokamaks
    - Spheromaks
    - Reversed Field Pinches
  - Strongly shaped cross sections:
    - Low aspect ratio, up-down asymmetry, high elongation ($\kappa \sim 6$)
    - Diverted or limiter equilibria
- **Code has most extensive set of diagnostics of any linear ideal code:**
  - Equilibrium and Eigenmode diagnostics
- **Code is easy to use and highly reliable:**
  - Extensive internal error checking
- **Major present drawback is speed:**
  - Continually improving
  - Plans to incorporate SuperLU parallel eigenvalue solver
    - $\Rightarrow$ Should yield orders of magnitude speedup

Continuing improvements will extend utility, accuracy and code performance in the future